THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixty-second regular meeting of the Society was held in New York City on Saturday, February 22, 1913. The attendance at the two sessions included the following thirty-eight members:

Professor R. C. Archibald, Mr. H. Bateman, Mr. R. D. Beetle, Mr. A. A. Bennett, Dr. Henry Blumberg, Professor Joseph Bowden, Professor E. W. Brown, Professor B. H. Camp, Professor A. B. Coble, Dr. Emily Coddington, Professor F. N. Cole, Dr. Elizabeth B. Cowley, Miss L. D. Cummings, Mr. C. H. Currier, Dr. L. L. Dines, Professor W. B. Fite, Mr. G. M. Green, Professor C. C. Grove, Professor C. N. Haskins, Mr. S. A. Joffe, Professor Edward Kasner, Dr. N. J. Lennes, Mr. P. H. Linehan, Professor James Maclay, Dr. R. L. Moore, Professor Frank Morley, Mr. F. S. Nowlan, Professor W. F. Osgood, Mrs. Anna J. Pell, Professor R. G. D. Richardson, Professor L. P. Siceloff, Mr. L. L. Smail, Professor D. E. Smith, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Professor W. A. Wilson.

Ex-President H. S. White occupied the chair, being relieved by Professors E. W. Brown and F. Morley. The Council announced the election of the following persons to membership in the Society: Professor E. P. Adams, Princeton University; Dr. H. L. Agard, Williams College; Professor Fiske Allen, Kansas State Normal School; M. Farid Boulad, Egyptian State Railways; Professor J. A. Caparo, Notre Dame University; Mr. C. H. Clevenger, Kansas State Agricultural College; Dr. A. L. Daniels, Jr., Yale University; Mr. W. Van N. Garretson, University of Michigan; Mr. G. M. Green, Columbia University; Mr. C. E. Love, University of Michigan; Dr. Thomas Muir, Education Office, Capetown, S. A.; Mr. J. A. Nyberg, University of Wisconsin; Dean Marion Reilly, Bryn Mawr College; Professor B. L. Remick, Kansas State Agricultural College; Professor W. V. Skiles, Georgia School of Technology; Mr. J. N. Vedder, University of Illinois. Five applications for membership in the Society were received.

The early publication was announced of the Princeton Colloquium Lectures, delivered at Princeton in 1909 by Professors G. A. Bliss and Edward Kasner.

The following papers were read at this meeting:

(1) Professor HARRIS HANCOCK: "A theorem in the analytic geometry of numbers."

(2) Professor B. H. CAMP: "The expression of a multiple integral as a simple integral."

(3) Mr. G. M. GREEN: "Projective differential geometry of triple systems of surfaces."

(4) Dr. C. A. FISCHER: "A generalization of Volterra's derivative of a function of a curve."

(5) Mr. L. B. ROBINSON: "Notes on the theory of systems of partial differential equations."

(6) Professor OSWALD VEBLEN and Mr. J. W. ALEXANDER, II: "Manifolds of n dimensions."

(7) Professor R. G. D. RICHARDSON: "Oscillation theorems for linear homogeneous self-adjoint partial differential equations with one parameter."

(8) Miss L. P. COPELAND: "Concerning the theory of invariants of plane n-lines."

(9) Dr. T. H. GRONWALL: "On the summability of Fourier's series."

(10) Dr. T. H. GRONWALL: "On Lebesgue's constants in the theory of Fourier's series."

(11) Dr. T. H. GRONWALL: "On the degree of convergence of Laplace's series."

(12) Dr. N. J. LENNES: "Note on Lebesgue and Pierpont integrals."

(13) Dr. N. J. LENNES: "Finite sets and the foundations of arithmetic."

(14) Mr. H. BATEMAN: "The expression of the equation of the general quartic curve in the form $A/xx' + B/yy' + C/zz' = 0$."

(15) Mr. H. BATEMAN: "Sonin's polynomials and their relation to other functions."

(16) Dr. DUNHAM JACKSON: "On the accuracy of trigonometric interpolation."

(17) Mr. C. E. WILDER: "On the degree of approximation to discontinuous functions by trigonometric sums."

(18) Professor EDWARD KASNER: "Systems of curves connected with equilong transformations."

Dr. Fischer was introduced by Professor Cole, Mr. Robinson by Dr. Cohen. Miss Copeland's paper was communicated to the Society through Professor Glenn, Mr. Wilder's through

Dr. Jackson. In the absence of the authors the papers of Professor Hancock, Miss Copeland, Dr. Gronwall, Dr. Jackson, and Mr. Wilder were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Hancock's paper is in abstract as follows: Let the coordinates of a point in three-dimensional space be denoted by (x, y, z), and further assume that these quantities take only positive integral values. A point is called a unit point if the greatest common divisor of its coordinates is unity. An asymptotic value is found for N, the number of unit points of a fixed rectangular parallelopipedon. It is then evident that the probability that a point is a unit point, when chosen at random among the points (x, y, z), where x varies from k to k_1, y varies from m to m_1 and z varies from n to n_1, is $N/(k_1 - k)(m_1 - m)(n_1 - n)$. In particular, if k, m, n remain fixed, while at least two of the numbers k_1, m_1, n_1 become very large, this probability is $\frac{6}{\pi^2}$ nearly.

2. Professor Camp's paper presents some general theorems, of which the following are corollaries:

1. If the function u is defined, limited, and Lebesgue integrable in the multiple, limited field A, and possesses the property that the set of points in A where u remains constant has measure zero, and if v is any other absolutely Lebesgue integrable function defined in A, then the following results hold: $U(x)$ and $V(x)$ exist in the interval $(0, a = \text{meas } A)$ so that, if $B_{U(x)}$ is the set in A where $u < U(x)$, $x = \text{meas } B_{U(x)}$, U is monotone increasing, and

$$\int_0^x U dx = \int_{B_{U(x)}} u dA, \quad \int_0^x V dx = \int_{B_{U(x)}} v dA, \quad \int_0^x UV dx = \int_{B_{U(x)}} uv dA.$$

2. If also $U(x)$ is an integra , ·

$$\int_A uv dA = U(a) \int_A v dA - \int_0^a \left(U'(x) \int_{B_{U(x)}} v dA \right) dx.$$

The theorems are useful in extending to multiple integrals theorems that have been established for simple integrals only. In particular this is true in the case of a certain theorem of Lebesgue (*Annales de la Faculté de Toulouse*, series 3, volume 1 (1909), page 65, v).

3. Mr. Green bases a projective theory of triple families of surfaces on the consideration of a certain completely integrable system of six partial differential equations of the second order, of which

$$y^{(k)} = f^{(k)}(u, v, w) \qquad (k = 1, 2, 3, 4) \tag{1}$$

are a fundamental system of solutions. Any fundamental system of solutions will then give a projective transformation of the system (1). The equations (1) give, for $u = \text{const.}$, $v = \text{const.}$, $w = \text{const.}$, the three families of surfaces, if the y's be interpreted as homogeneous coordinates of a point in space. The transformations

$$\begin{gathered} y = \lambda(u, v, w)\bar{y}, \\ \bar{u} = U(u), \qquad \bar{v} = V(v), \qquad \bar{w} = W(w) \end{gathered} \tag{2}$$

are the most general transformations which leave unchanged the triple family of surfaces, without interchanging the families. According to the general method of Professor Wilczynski, the invariants and covariants of the system of differential equations under the transformations (2) are calculated; the geometric interpretation of these invariants and covariants constitutes a projective theory of the triple system.

4. Volterra has defined the derivative of a function of a curve at a point, and has proved that, if it satisfies certain conditions, the first variation of the function $L(C)$ can be expressed by the equation

$$\delta L(C) = \int_{x_1}^{x_2} L'(C, t)\, \delta y(t)\, dt,$$

where $L'(C, t)$ is the derivative of $L(C)$ at the point $x = t$, and where x_1 and x_2 are the end points of the curve C whose equation is $y = y(x)$. In the present paper Dr. Fischer has considered only the class of curves which give fixed end values to a set of m functions determined by m ordinary differential equations of the first order. The definition of the derivative is modified so as to apply to functions defined for this class of curves; the theorem mentioned above is proved in a slightly different form, and its application to the Lagrange problem of the calculus of variations is discussed.

5. Mr. Robinson discusses the systems of partial differential equations named by Riquier regular. An error made by Delassus in his extension of the theorem of Cauchy is corrected. Likewise the author considers the passivity conditions of systems of partial differential equations and shows how some of Riquier's rules can be extended and simplified.

6. The paper of Professor Veblen and Mr. Alexander contains a discussion of n-dimensional manifolds by means of matrices reduced modulo two. Certain theorems are proved which reduce to known results for two-sided manifolds, but which are new for one-sided manifolds. The paper will appear in the June number of the *Annals of Mathematics*.

7. By means of his theory of integral equations Hilbert has proved the existence of functions $u(x, y)$ vanishing on the boundary of a region R and satisfying within that region the most general linear self-adjoint partial differential equation of the second order in two variables and containing one parameter

$$(pu_x)_x + (pu_y)_y + q(x, y)u + \lambda k(x, y)u = 0, \quad (p(x, y) > 0).$$

In the orthogonal case ($k \geqq 0$) there are an infinite number of parameter values $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots$ for which such solutions exist; in the polar case (k both signs, $q \leqq 0$) there are two infinite sets $0<\lambda_1\leqq\lambda_2\leqq\lambda_3\leqq\cdots$; $0>\lambda_{-1}\geqq\lambda_{-2}\geqq\lambda_{-3}\geqq\cdots$. Professor Richardson has undertaken the investigation of the nature of the various solutions and shows that to a given integer n there correspond in the orthogonal case at least one, in the polar case at least two solutions $u(x, y)$ of the type sought and such that there are n sub-regions of R on the boundary of which each solution also vanishes. After deriving an existence theorem for the non-orthogonal non-polar equation (k both signs, q positive in at least a portion of R), he shows that in this case there is an integer $n_1 \geqq 0$ such that for $n < n_1$ there is no solution of the type sought, while for $n \geqq n_1$ there are at least two.

This discussion, which holds also for equations in three or more independent variables, will appear in the *Mathematische Annalen*.

8. In the first section of Miss Copeland's paper she establishes necessary and sufficient conditions in order that two

factorable ternary forms f_{2m}, g_{2m}, representing m-lines, may have the property that each form is the sum of constants times the m mth powers of the linear factors of the other. In the case where the simplest full invariant of each m-line vanishes it is shown that the plane pencils have a common vertex and are apolar if they have the above property.

In the second part of the paper the general theory of full invariants is studied. The necessity and sufficiency of the ternary annihilators is established by means of symmetric functions and the solution of linear partial differential equations. This is generalized for n variables. The subject of complete systems is treated by the direct extension of the process for binary forms due to Hilbert.* Complete systems of invariants for a triangle, quadrilateral, and pentagon are obtained, and of covariants for a triangle and quadrilateral. It is shown how the invariants may be expressed rationally in terms of the $2m$ independent coefficients of the m-line's form.

9. Dr. Gronwall gives a simplified proof of the theorem due to Riesz and Chapman, that the Fourier series of an absolutely integrable function $f(x)$ is summable by Cesàro's means of order $k > 0$ with the sum $\frac{1}{2}\lim_{\epsilon=0}(f(x+\epsilon)+f(x-\epsilon))$ at any point where this limit exists, and shows by an example that the theorem is not generally true for a function which is integrable without being absolutely integrable.

10. Dr. Gronwall has shown (*Mathematische Annalen*, volume 72, 1912) that the Lebesgue constants ρ_n increase with n, beginning with $n = 1$; and an independent proof was given by Jackson (*Transactions*, 1912); both proofs depend upon the representation of ρ_n by a definite integral. The present paper gives a proof based directly on Fejér's explicit trigonometrical formula for ρ_n, which is preferable from a systematic point of view.

11. In the present paper, Dr. Gronwall considers the Laplace development in a series of spherical harmonics of a function $f(\theta, \varphi)$ of the geographical coordinates on the unit sphere under the assumption that for any two points θ, φ and θ', φ'

$$|f(\theta, \varphi) - f(\theta', \varphi')| < \omega(\delta),$$

* *Math. Annalen*, vol. 33.

where ω is a given function and δ the distance between the two points. It is shown that the remainder after n terms in Laplace's series for $f(\theta, \varphi)$ is numerically less than a constant multiple of $\omega(1/n)\sqrt{n}$, and that there exist functions $f(\theta, \varphi)$ such that at a given point the remainder, for an infinite number of values of n, is numerically greater than $\Omega(1/n)\sqrt{n}$, where Ω is any function such that $\lim_{\delta=0} \Omega(\delta)/\omega(\delta) = 0$.

In the particular case of the Legendre series for a function $f(x)$, $-1 \leqq x = \cos\theta \leqq 1$, these results apply to the end points $x = \pm 1$ and thus complete the investigations of Dr. Jackson (*Transactions*, 1912) which apply to any interval $-1+\epsilon \leqq x \leqq 1-\epsilon$, $(\epsilon > 0)$. As in his theorem $\sqrt{n}$ is replaced by $\log n$, it is seen that there is an essential difference between end points and interior points in Legendre's series (which does not exist, for instance, in the Fourier series).

12. In this paper Dr. Lennes compares the definitions of definite integrals given by Lebesgue and Pierpont. Let c denote the continuum on a certain interval ab, and d any subset of c on this interval. Let $c - d = e$. It is shown that if the function is defined on a measurable set d (in the Lebesgue sense) the two definitions are equivalent. If the set d is not measurable the Lebesgue definition does not apply while that of Pierpont does. However in this case the Pierpont integral does not satisfy one of the fundamental requirements, viz.,

$$\int_d f + \int_e f = \int_c f.$$

By adopting a modification of Pierpont's definition which limits its applicability to the field covered by Lebesgue's, it is possible to make the treatment simpler than Lebesgue's.

13. Dr. Lennes' second paper gives a set of assumptions for point sets and from them derives the usual assumptions for arithmetic.

14. In Mr. Bateman's first paper it is shown that the equation of a general quartic curve can be expressed in the form

$$Ayy'zz' + Bzz'xx' + Cxx'yy' = 0,$$

where $xx'yy'zz' = 0$ is the equation of six straight lines. For

a certain type of quartic curve this reduction can be effected in an infinite number of ways.

15. The polynomials introduced by Sonin are discussed in Mr. Bateman's second paper. Various definite integrals and expansions are obtained and the polynomials are used to obtain some elementary solutions of the equation of wave motion.

16. If the values of the function $f(x)$, of period 2π, are known at $2n+1$ points equally spaced throughout a period, a trigonometric sum of order n at most which takes the same values as $f(x)$ at these points can be constructed by means of formulas analogous to those which define the Fourier series of $f(x)$. The question of the convergence of this trigonometric sum to the value $f(x)$, when the number n is indefinitely increased, has been investigated by Faber (*Mathematische Annalen*, volume 69). Dr. Jackson points out that his theorems, recently communicated to the Society, on the accuracy of trigonometric approximation, in conjunction with a lemma proved by Faber, immediately furnish information as to the rapidity of the convergence of the interpolation formula, the results being similar to those obtained in the case of Fourier's series. For example, if $f(x)$ satisfies a Lipschitz condition, the error does not exceed a constant multiple of $(\log n)/n$. A somewhat less simple formula, still determined by a finite number of values of $f(x)$, is found to give an error not exceeding a quantity of the order of $1/n$, when $f(x)$ satisfies a Lipschitz condition. This formula has a further advantage with reference to the possible effect of errors of observation, if the values of $f(x)$ used are subject to such errors.

17. In this paper Mr. Wilder shows that any function $f(x)$, of period 2π, which has in any finite interval no other discontinuities than a finite number of finite jumps, and in any interval not including one of these points of discontinuity satisfies a Lipschitz condition

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

is approximately represented, at any point x whose distance from the nearest point of discontinuity is at least as great as δ:

1. By means of a certain trigonometric sum of order n at most, with an error not exceeding

$$\frac{1}{n}\left(c_1\lambda + c_2\frac{\nu}{\delta}\right),$$

where c_1 and c_2 (like $c_3, \cdots, c_6$ below) are absolute constants, and ν is the difference between the upper and lower limits of $f(x)$.

2. By means of Fejér's arithmetic mean of the first $n+1$ terms ($n > 1$) of the Fourier series of $f(x)$, with an error not exceeding

$$\frac{\log n}{n}\left(c_3\lambda + c_4\frac{\nu}{\delta}\right).$$

3. By means of the first $n+1$ terms ($n > 1$) of the Fourier series itself, with an error not exceeding

$$\frac{\log n}{n}\left(c_5\lambda + c_6\mu\frac{\nu}{\delta}\right),$$

where μ is the number of discontinuities in any interval of length 2π.

18. The systems of curves studied by Professor Kasner play (roughly) the same role in the geometry of the dual variable $u + jv$ ($j^2 = 0$) as the isothermal systems in the geometry of the ordinary complex variable $x + iy$ ($i^2 = -1$). The analogy is not complete, since the Laplace equation $\psi_{xx} + \psi_{yy} = 0$ is replaced by the simpler equation $\psi_{vv} = 0$, but a list of analogous properties (including new results for the isothermal type) is obtained.

F. N. COLE,
Secretary.

THREE OR MORE RATIONAL CURVES COLLINEARLY RELATED.

BY DR. JOSEPH E. ROWE.

(Read before the American Mathematical Society, December 31, 1912.)

Introduction.

THE R^n, or rational plane curve of order n, possesses certain sets* of covariant rational point and line curves which

* J. E. Rowe, "Bicombinants of the rational plane quartic and combinant curves of the rational plane quintic," *Transactions*, vol. 13 (July, 1912), pp. 388–389.

are related in the following manner: if A, B, and C are three rational point loci of such a set, any parameter value substituted in the parametric equations of these curves yields the coordinates of three collinear points; similarly, if A', B', and C' are three line loci of such a set, any parameter value substituted in their parametric equations yields the coordinates of three concurrent lines. Curves related in either of the above ways may be called *collinearly related*, or curves *in collinear relation*. It is not remarkable that curves so related could be found by constructing their equations algebraically, but it is worthy of notice that such sets of curves naturally arise as covariants of rational plane curves. This fact has received no attention in mathematical literature, and it is desirable, therefore, to give several examples of such sets of curves, to outline the conditions which give rise to them, and to indicate some analogous facts for curves of hyperspace.

§ I *a*. *Genesis of Covariant* R^k *of* R^n.

Let the R^n be written parametrically

$$x_0 = (\alpha t)^n, \qquad x_1 = (\beta t)^n, \qquad x_2 = (\gamma t)^n. \tag{1}$$

The rth osculant of the R^n at a point whose parameter is t' is an R^{n-r} whose parametric equations are

$$x_0 = (\alpha t)^{n-r}(\alpha t')^r, \qquad x_1 = (\beta t)^{n-r}(\beta t')^r, \qquad x_2 = (\gamma t)^{n-r}(\gamma t')^r. \tag{2}$$

Suppose the R^{n-r} has a covariant line whose equation is

$$k_0 x_0 + k_1 x_1 + k_2 x_2 = 0, \tag{3}$$

in which the k's are functions of degree p' in the coefficients of t which occur in the parametric equations of the R^{n-r}. If (3) is actually calculated for (2) and t' made equal to t (to signify that it has become variable) the result will be a binary p-ic in $t (p = rp')$ which may be written

$$(\alpha' t)^p x_0 + (\beta' t)^p x_1 + (\gamma' t)^p x_2 = 0. \tag{4}$$

The envelope of the line (4) is a rational curve of class p whose parametric equations are

$$\zeta_0 = (\alpha' t)^p, \qquad \zeta_1 = (\beta' t)^p, \qquad \zeta_2 = (\gamma' t)^p. \tag{5}$$

This curve is a covariant rational curve of the R^n and its point equations may be found from (5).

§ I *b*. *The Rôle of Polars.*

The line section of (1) by a line

$$\zeta_0 x_0 + \zeta_1 x_1 + \zeta_2 x_2 = 0 \tag{6}$$

may be written symbolically

$$(a\zeta)(\pi t)^n = 0, \tag{7}$$

and the binary $(2n-2)$-ic whose roots are the parameters of the $2n-2$ tangents from a point (6) may be similarly written

$$(Ax)(\delta t)^{2n-2} = 0. \tag{8}$$

There are sets of parameters on the R^n which are covariant; i. e., the binary form of which they are roots is unaltered by a linear transformation of the x's or t's. For instance, the parameters of the $3n-6$ flexes, or the $(n-1)(n-2)$ nodal parameters, as well as the set of parameters cut out of R^n by any covariant locus constitute such a set. Suppose such a set is given by

$$(ct)^q = 0. \tag{9}$$

The polar of (9) with respect to (7) or of (7) with respect to (9) according as $q > n$ or $< n$ yields a covariant rational point curve of R^n of degree $q-n$ or $n-q$. If $n = q$, the apolarity condition of (7) and (9) is the equation of a covariant point of the R^n. Similarly the polar of (8) with respect to (9) or of (9) with respect to (8) is the equation of a covariant rational line curve of degree $2n-2-q$ or $q-2n+2$, according as $2n-2 > q$ or $< q$. If $2n-2 = q$, the apolarity condition is the equation of a covariant line of the R^n. Several illustrations will bring out these facts more clearly.

§ II *a*. *A Set of Collinear Conics.*

Let the R^4 be written

$$x_i = a_i t^4 + b_i t^3 + c_i t^2 + d_i t + e_i \quad (i = 0, 1, 2). \tag{10}$$

The parameters of its six flexes* are the roots of

* J. E. Rowe, "Important covariant curves and a complete system of invariants of the rational quartic curve," *Transactions*, vol. 12 (July, 1911), p. 299.

$$\alpha' t^6+3\lambda' t^5+(3\beta'+6n')t^4+(\gamma+8\delta)t^3+(3\beta+6n)t^2 +3\lambda t+\alpha=0. \tag{11}$$

The line section of 10 by the line (6) is

$$(a\zeta)t^4+(b\zeta)t^3+(c\zeta)t^2+(d\zeta)t+(e\zeta)=0; \tag{12}$$

the polar quadratic of (12) with respect to (11) multiplied by 20 is

$$\begin{aligned} L_\rho = {} & [4(a\zeta)(\beta+2n)-(b\zeta)(\gamma+8\delta)+4(c\zeta)(\beta'+2n') \\ & -10(d\zeta)\lambda'+20(e\zeta)\alpha']t^2 \\ & +[20(a\zeta)\lambda-8(b\zeta)(\beta+2n)+2(c\zeta)(\gamma+8\delta) \\ & -8(d\zeta)(\beta'+2n')+20(e\zeta)\lambda']t \\ & +[20(a\zeta)\alpha-10(b\zeta)\lambda+4(c\zeta)(\beta+2n) \\ & -(d\zeta)(\gamma+8\delta)+4(e\zeta)(\beta'+2n')]=0. \end{aligned} \tag{13}$$

This is a line section of the conic* whose parametric equations are

$$\begin{aligned} x_i = {} & [4a_i(\beta+2n)-b_i(\gamma+8\delta)+4c_i(\beta'+2n')-10\delta_i\lambda' \\ & +20e_i\alpha']t^2 \\ & +[20a_i\lambda-8b_i(\beta+2n)+\ .\ .\ .\ ]t \\ & +[20a_i\alpha-10b_i\lambda+\ .\ .\ .\ ] \qquad (i=0,1,2). \end{aligned} \tag{14}$$

A line cuts the R^4 and (14) in six points whose parameters are apolar to (11).

The parameters of a covariant pair of points on the R^4 are given by the quadratic

$$(\beta'-3n')t^2+(\gamma-2\delta)t+(\beta-3n)=0. \tag{15}$$

The polar quadratic of (15) with respect to (12), multiplied by 12, may be written in the form

$$\begin{aligned} L_s = {} & [2(c\zeta)(\beta'-3n')+3(b\zeta)(2\delta-\gamma)+12(a\zeta)(\beta-3n)]t^2 \\ & +[6(d\zeta)(\beta'-3n')+4(c\zeta)(2\delta-\gamma)+6(b\zeta)(\beta-3n)]t \\ & +12(e\zeta)(\beta'-3n')+3(d\zeta)(2\delta-\gamma)+2(c\zeta)(\beta-3n)]=0. \end{aligned} \tag{16}$$

A line section of (16) together with (15) form a binary quartic

* We call the discriminant of (13) P; the same function of (16) S.

which is apolar to the binary quartic defining collinear points of R^4.

The flex lines of all first osculants of the R^4 envelope a conic* N; if L_N is a line section of this conic we find that

$$2(L_\rho - 2L_s) = 5L_N. \tag{17}$$

Also, the flex tangents of all first osculants form triangles whose vertices are on a conic† B; if L_B is a line section of this conic,

$$9L_\rho + 2L_s = 2L_B. \tag{18}$$

Of these four conics (14), (16), and B may be found by the method described in § I *b*; the conic N may be derived as the curves in § I *a*.

For convenience in writing let the equations of P in (14) be written

$$x_i = P_{0i}t^2 + P_{1i}t + P_{2i}, \tag{19}$$

and those of S, which may be read off from (16), be put in the form

$$x_i = s_{0i}t^2 + s_{1i}t + s_{2i}. \tag{20}$$

By reason of (17), (18), (19) and (20) the parametric equations of N and B are

$$x_i = (P_{0i} - 2s_{0i})t^2 + (P_{1i} - 2s_{1i})t + (P_{2i} - 2s_{2i}), \tag{21}$$

and

$$x_i = (9P_{0i} + 2s_{0i})t^2 + (9P_{1i} + 2s_{1i})t + (9P_{2i} + 2s_{2i}) \tag{22}$$

$$(i = 0, 1, 2.)$$

Suppose the coordinates of the points whose parameter on each of these is t_1 are found by substituting $t_1 = t$ in equations (19)–(22). If the coordinates of this point t_1 on (19) are π_i, and those of the point t_1 on (20) are Σ_i, the coordinates of the points on N and B with this same parameter value are $\pi_i - 2\Sigma_i$ and $9\pi_i + 2\Sigma_i$. *Hence, the points which have the same parameter value on the four conics P, S, N and B are collinear.* Evidently this relation is true for all conics whose line sections are of the form

$$aL_\rho + bL_s + cL_N = 0. \tag{23}$$

* W. Stahl, "Ueber die rationale ebene Curve vierter Ordnung," *Crelle*, vol. 101 (1887), p. 314.

† Loc. cit., p. 306.

§ II *b*. *A Set of Collinear Line Curves.*

As a second illustration we may take a set of covariant rational line curves of the R^5 which introduce an interesting new feature.

In (1), let $n = 5$. The first osculant of R^5 at t' is

$$x_0 = (\alpha t)^4(\alpha t'), \quad x_1 = (\beta t)^4(\beta t'), \quad x_2 = (\gamma t)^4(\gamma t'). \tag{24}$$

The R^4 possesses a pencil of covariant lines whose properties have been discussed in a previous paper.* These lines are of degree eleven in the coefficients of the equations of the R^4. If this pencil of lines is

$$\lambda_1 L_3 + \mu_1 L_4 = 0 \tag{25}$$

and these are actually calculated for (24), the result is a system of rational curves of class eleven such that the tangents to any three at a point whose parameter is given are concurrent lines. The proof of this fact is exactly similar to the proof of the theorem in the last section, both being a matter of linear dependence. In fact, all rational curves of class eleven formed in this way by assigning particular values to μ_1/λ_1 are related in the same manner.

In particular if $\lambda_1 = -1$ and $\mu_1 = 5$ in (25), the result is not another line whose coefficients are of degree eleven in the coefficients of the parametric equations of the R^4, but the product of an invariant of the R^4 of order six in these coefficients and a line L_1 whose coefficients are of degree five in the coefficients of the R^4. The invariant just mentioned calculated for (24) yields a binary sextic which equated to zero has roots which are a set of covariant parameters on the R^5. The line L_1 for (24) envelopes a rational curve of class 5. Hence, *if a particular value of t is substituted in the equations of any two eleventhics derived from* (25) *and in the equations of the rational quintic just mentioned, the results are the coordinates of three concurrent lines.*

Similarly, the R^n could have three covariant rational line or point curves, all of different degrees, collinearly related. If C_α, C_β, C_γ are three line sections or point projections of three covariant rational point or line curves of R^n, then this set of covariant curves of R^n is in collinear relation if only a relation of the form

$$\lambda I_1 C_\alpha + \mu I_2 C_\beta + \nu I_3 C_\gamma = 0$$

* *Transactions*, vol. 13 (1912), pp. 388–389.

exists, where λ, μ, and ν are constants and I_1, I_2, and I_3 are invariants of the R^n.

§ III. *Extension to Space.*

The processes which have been outlined and illustrated for rational plane curves can be extended directly to space of higher dimensions. Let the R_m^n denote a rational curve of order n in space of m dimensions. A covariant linear form of an R_m^{n-r} in x or ξ, by the use of osculants, gives rise to a covariant rational class or order curve of the R_m^n. The existence of covariant sets of parameters on R_m^n and the use of polars makes possible the derivation of other covariant rational curves of the R_m^n. The criterion for the existence of collinear curves, as in the case of plane curves, is merely a matter of linear dependence but the number of varieties of collinearity will increase with m. For instance, two varieties of collinearity are possible when $m = 3$ and these may be illustrated as follows: First, let the R_3^n have a set of covariant rational curves whose parametric point* equations are given, and let A, B, C, and D be the sections of four curves of the set by a plane; if a relation of the form

$$\lambda A + \mu B + \nu C = D$$

exist, in which λ, μ, and ν are constants, a parameter substituted in the parametric point equations of these four curves yields the coordinates of four coplanar points. Second, let the R_3^n have a set of covariant rational curves whose parametric point equations are known, and let A', B', and C' be the sections of three of these curves by a plane; if a relation of the form

$$\lambda' A' + \mu' B' = C'$$

exist, in which λ' and μ' are constants, any parameter substituted in the parametric point equations of these curves yields the coordinates of three points on a line.

DARTMOUTH COLLEGE,
November, 1912.

* I have inserted the word "point" because it is desirable to emphasize the fact that these curves are to be considered loci of a point.

SECOND NOTE ON FERMAT'S LAST THEOREM.

BY PROFESSOR R. D. CARMICHAEL.

In a note printed on pages 233–236 of the present volume of the BULLETIN I have proved the following theorem:

If p is an odd prime and the equation

$$x^p + y^p + z^p = 0$$

has a solution in integers x, y, z each of which is prime to p, then there exists a positive integer s, less than $\frac{1}{2}(p-1)$, such that

$$(1) \qquad (s+1)^{p^2} \equiv s^{p^2} + 1 \mod p^3.$$

Professor Birkhoff has called my attention to the fact that condition (1) may be replaced by the simpler condition

$$(1') \qquad (s+1)^p \equiv s^p + 1 \mod p^3,$$

these two conditions being equivalent. Let us define the integers λ and μ by the relations

$$(s+1)^p = s + 1 + \lambda p, \quad s^p = s + \mu p.$$

Then

$$(2) \qquad (s+1)^p = s^p + 1 + (\lambda - \mu)p.$$

We have also

$$\begin{aligned}(s+1)^{p^2} &\equiv (s+1)^p + \lambda p^2 (s+1)^{p-1} \mod p^3 \\ &\equiv s + 1 + \lambda p + \lambda p^2 \mod p^3 \\ &\equiv s + 1 + \lambda(p + p^2) \mod p^3.\end{aligned}$$

Likewise

$$s^{p^2} \equiv s + \mu(p + p^2) \mod p^3.$$

From the last two congruences we have

$$(3) \qquad (s+1)^{p^2} \equiv s^{p^2} + 1 + (\lambda - \mu)(p + p^2) \mod p^3.$$

From (2) and (3) we see that a necessary and sufficient condition for either (1) or (1') is that $\lambda - \mu \equiv 0 \mod p^2$. Therefore (1) and (1') are equivalent.

The simpler relation (1') can be derived more readily than the relation (1). For from the congruence $x + y + z \equiv 0 \mod p^2$, obtained in my previous paper, we have immediately $(x+y)^p \equiv -z^p \mod p^3$. Hence

$$(x+y)^p \equiv x^p + y^p \mod p^3,$$

from which (1′) is readily deduced.

Professor Birkhoff points out further that the test fails to be effective for all primes p of the form $6n+1$. For if $p = 6n+1$ it follows from the theory of primitive roots modulo p^3 that the congruence

$$t^6 \equiv 1 \mod p^3$$

has a solution t for which $t-1$ is prime to p. Hence also

$$t^2 + t + 1 \equiv 0 \mod p^3.$$

Then we have

$$(t+1)^p = (t+1)(t+1)^{6n} \equiv (t+1)(-t^2)^{6n} \equiv t+1 \mod p^3,$$

$$(t+1)^{p^2} \equiv (t+1)^p \equiv t+1 \mod p^3,$$

and

$$t^p \equiv t \cdot t^{6n} \equiv t \mod p^3, \qquad t^{p^2} \equiv t^p \equiv t \mod p^3:$$

Therefore

$$(t+1)^{p^2} \equiv t^{p^2} + 1 \mod p^3.$$

Now put

$$t = \sigma + vp, \qquad (0 < \sigma < p-1).$$

Then

$$t^{p^2} \equiv \sigma^{p^2}, \quad (t+1)^{p^2} \equiv (\sigma+1)^{p^2} \mod p^3.$$

Therefore

$$(\sigma+1)^{p^2} \equiv \sigma^{p^2} + 1 \mod p^3, \quad (0 < \sigma < p-1).$$

This is relation (7) of my previous note; from this follows (1) as in the earlier treatment. Hence (1) is satisfied by all primes of the form $6n+1$. Therefore the test can be useful only when the exponent p is 3 or is of the form $6n-1$.

INDIANA UNIVERSITY,
March, 1913.

AN EXTENSION OF A THEOREM OF PAINLEVÉ.

BY DR. E. H. TAYLOR.

(Read before the American Mathematical Society, October 26, 1912.)

THEOREM: Let $f(z)$ be a function which is single-valued and analytic throughout the interior of a region S of the z-plane, $z = x + yi$. If $f(z)$ vanishes at every point of a

connected portion of the boundary of S, two points of which, A and B, can be joined by a curve C lying wholly within S, then $f(z) \equiv 0$.

Painlevé* has proved this theorem for the case where the portion of the boundary in question is an arc of a regular curve. The object of the present paper is to show that the theorem is true in the general case for which the theorem is above stated; for example, when in every neighborhood of any point of the boundary there are points that cannot be approached along a continuous curve lying in the region.

Denote by Σ the region bounded completely by C and the portion AB of the boundary along which $f(z)$ vanishes. We will assume that Σ lies within a circle of radius unity with center at O: $(z = 0)$, a point of AB, but which is distinct from both A and B.

Let the z-plane be transformed by

$$z_1 = \log z. \tag{1}$$

The interior of the unit circle is thus mapped on the half-plane $x_1 < 0$, Σ going over into a region Σ_1 of that half-plane which extends to infinity and has the point $z_1 = 0$ as an exterior point.

We will next apply the transformation

$$z_2 = -i/z_1. \tag{2}$$

The interior of Σ_1 is thereby carried over into the interior of a region Σ_2 which lies in the upper half of the z_2-plane. Thus Σ has been mapped on Σ_2, the boundary point O: $(z = 0)$ of Σ going over into the boundary point $z_2 = 0$ of Σ_2.

As a third transformation we will use

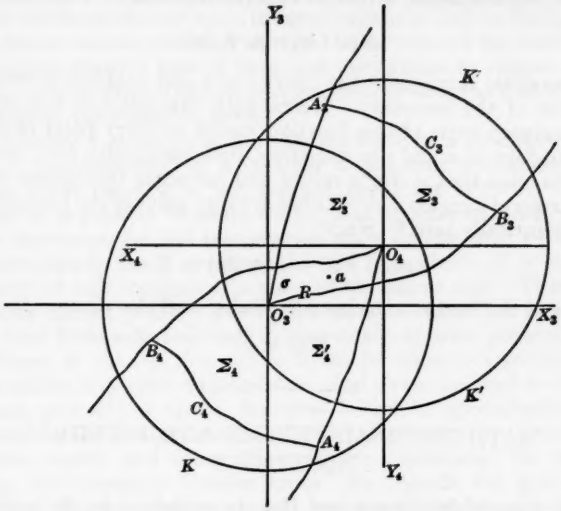
$$z_3 = \sqrt{z_2}, \quad z_3 = r_3 e^{\varphi_3 i}, \tag{3}$$

which makes the image Σ_3 of Σ lie within the region $0 < \varphi_3 < \pi/2$, as indicated in the figure. We will denote the images in the z_3-plane of O, A, B, and C by O_3, A_3, B_3, and C_3, respectively.

There is a neighborhood of O in the original region Σ that contains no boundary point of Σ in which $f(z)$ does not vanish; in particular, this neighborhood contains no point of C.

* *Toulouse Annales*, vol. 2 (1888), p. B 29.

Therefore, with O_3 as center, it is possible to construct a circumference K outside of which C_3 will lie, and which will cut off from Σ_3 one or more regions. One of these regions Σ_3' will



contain a point a in its interior whose distance from O_3 is less than half the radius of K.

Next, rotate the plane about the point a as a center by the transformation

$$z_4 = -z_3 + 2a. \tag{4}$$

This transforms the axes of reals and pure imaginaries into lines parallel to them, and bounding with them a rectangle R.

Let the images in the z_4-plane of Σ_3' and K be called Σ_4' and K'. The regions Σ_3' and Σ_4' have one or more regions in common, all of which lie in R, and hence in the region common to K and K'. Let σ denote one of these regions. From the method of constructing K it follows that K, and hence K', contain the images of no boundary points of Σ except those in which $f(z)$ vanishes. Consequently every boundary point of σ is the image of a boundary point of Σ in which $f(z)$ vanishes.

Let the function into which $f(z)$ is carried by the transfor-

mations (1), (2), and (3) be denoted by $\varphi(z_3)$. The latter function is transformed by (4) into $\varphi(-z_3+2a)$. From the hypotheses of the theorem and the properties of the transformations employed, it follows that the function

$$\varphi(z_3)\cdot\varphi(-z_3+2a)$$

is analytic throughout the interior of σ and vanishes at every point of the boundary. Hence both the real and the pure imaginary parts of this function vanish at every point of the boundary of σ and are, therefore, both identically zero, since a function that is single-valued and harmonic throughout the interior of a region and vanishes at every point of the boundary is identically zero.* Since

$$\varphi(z_3)\cdot\varphi(-z_3+2a)\equiv 0,$$

one of the factors vanishes identically, and therefore

$$f(z)\equiv 0.$$

MATHEMATICAL PHYSICS AND INTEGRAL EQUATIONS.

Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik. Vorlesungen an der Universität zu Breslau, gehalten von ADOLF KNESER. Braunschweig, Vieweg, 1911. 8vo. viii+243 pp.

THE solution of various boundary value problems for a partial differential equation by means of the expansion of an arbitrary function in series of solutions of ordinary differential equations involving a parameter constitutes one of the most important applications of the theory of integral equations. Here, as so often elsewhere, mathematical physics has first propounded the question, and it has been the task of analysis to furnish the answer. Especially close, therefore, has been the connection between mathematical physics and integral equations; especially interesting must be likewise a method of treatment which aims to exhibit this connection as vividly as possible. Such is the method of Kneser's book; we learn

* Osgood, Lehrbuch der Funktionentheorie, vol. 1, 2d ed., 1912, p. 623.

from the preface that the author deals particularly with the applications, and makes the least possible appeal to the general theory. An idea of the fidelity of adherence to this plan may be obtained from the titles of the chapters: integral equations and the linear flow of heat; integral equations and oscillations of linear systems of masses; integral equations and the Sturm-Liouville theory; flow of heat and oscillations in regions of two or three dimensions; existence theorems and the Dirichlet problem; the Fredholm series.

Chapter I studies the temperature state in a straight rod or a ring immersed in a medium of constant temperature. The Green function is not dragged in by the heels for the sake of a possible ultimate utility, but appears naturally as an expression for the temperature state independent of the time (steady flow), produced by the imposition of a heat source of unit strength at a point of the bar or ring. That a source cannot produce steady flow in a rod when the escape of heat from side and ends is prevented, appears physically evident, so that in this case a Green function is impossible; considerations again of purely physical character lead to the usual generalized Green function. Various specializations yield, for the straight rod, the Fourier sine series, the Fourier cosine series, and other trigonometric expansions; for the ring, the complete Fourier series. As regards the general theory, this first chapter already contains the theorem on the expansion of an arbitrary function in terms of the principal solutions for a real symmetric kernel—much of the work, however, being based on the as yet unproved assumption that for such a kernel there is at least one principal parameter value.

In Chapter II we re-discover several formulas of the previous chapter, now clothed with a mechanical instead of a thermodynamic interpretation. The Green function is the displacement produced by a force function which is zero except at a single point. For this concept the author claims neither physical accuracy nor mathematical meaning a limit process may be used to clarify matters. The vibrating string leads us anew to the Fourier sine series. The transverse oscillations of a freely suspended heavy cord bring a result not previously obtained—the expansion in Bessel's functions of order zero; the work is momentarily only formal, as the necessary convergence theorems are postponed to the following chapter.

Indications are given for the similar treatment of a weightless cord rotating about an axis perpendicular to itself, leading to expansions in zonal harmonics (Legendre polynomials). The general theory is enriched by the first appearance of the resolvent function to the kernel of an integral equation.

The general Sturm-Liouville equation arrives in the next chapter, by way of the flow of heat in a non-homogeneous rod; the results of Chapter I are verified and extended. A lacuna is filled, for the cases in hand, by a proof that the kernel corresponding to a Sturm-Liouville equation gives an infinite number of real principal values for the parameter. Bessel's functions and zonal harmonics, which satisfy differential equations whose coefficients are not bounded, and which therefore escape the Sturm-Liouville theory, receive independent treatment. The existence of a unique solution of a linear differential equation of the second order for given values of function and first derivative is proved by the familiar machinery of successive approximations; the real gist of this work is an existence theorem, not (except indirectly) for a differential equation, but for an integral equation of the Volterra type,—the very form of the approximations emphasizes this.* The proof thrown into this form would have been more in keeping with the subject matter of the book, and would have been especially welcome in view of the fact that no other problem discussed by the author leads to a Volterra equation.

A new chapter extends the previous results to the plane and to space. Many interesting problems are solved; the only new point for the theory is the solution of an integral equation with discontinuous kernel by considering instead an equation with properly chosen iterated kernel. The author asserts (in other notation) of the Green function $K(\xi, \eta; x, y)$ for Laplace's equation that

$$K(\xi, \eta; x, y) = -\frac{1}{2\pi} \log \sqrt{(x-\xi)^2 + (y-\eta)^2} + M(\xi, \eta; x, y),$$

where M is a function of ξ, η, x, y continuous throughout the region. If this is understood to imply that ξ, η, x, y range independently over the region, the statement is not quite

* Cf. Mason, New Haven Mathematical Colloquium, p. 176.

accurate; E. E. Levi has shown* that although for common approach of (ξ, η) and (x, y) to the same *interior* point the discontinuity of K is completely characterized by the logarithmic term of the preceding formula, for common approach to a *boundary* point the discontinuity is of the nature of twice this logarithmic term.

In the next chapter we find a general proof for the existence of principal solutions for any continuous real symmetric kernel. Explicit formulas are given for the successive principal values; it is interesting to compare here the entirely different expressions obtained for the same purpose by I. Schur.† The author discusses, by a method due to Schmidt, the theorem that solutions of the homogeneous equation with unsymmetric kernel and solutions of the "transposed" equation occur simultaneously; but Kneser's presentation is incomplete. It is shown that solutions for $K(x, y)$ and for another kernel, which we shall call $K_1(x, y)$, do occur simultaneously; as $K_1(x, y)$ is of such form as to render it evident that $K_1(x, y)$ and $K_1(y, x)$ possess or fail to possess solutions simultaneously, the author regards the theorem as proved. The fallacy lies in the fact that $K_1(y, x)$ is not the same as the kernel $K_2(y, x)$ derived from $K(y, x)$ by the corresponding steps to those which evolved $K_1(x, y)$ from $K(x, y)$; Schmidt's own treatment completed the proof by showing that solutions of $K_1(y, x)$ and $K_2(y, x)$ occur simultaneously.

Several further applications to physical problems close this chapter.

At last, in Chapter VI, we come to the direct mathematical treatment of the integral equation. The case in which the parameter does not take a principal value is studied by use of the Fredholm functions $D(\lambda)$, $D(x, y; \lambda)$; the proof given for the Hadamard determinant theorem is elegant in its closeness to the geometric meaning of the theorem. The existence of at least one principal value for a real symmetric kernel is proved again, by a method due to Kneser himself; it is also shown that all poles of the resolvent function are simple. The book closes with a proof that the order of a root of the determinant for any real symmetric kernel having only

* For Green's functions of the second kind; *Göttinger Nachrichten* (1908), p. 248. The fact had been noted, at least for special forms of regions, by earlier writers.

† *Math. Annalen*, vol. 67 (1909), p. 306.

positive principal values (the latter restriction, however, is not essential for the truth of the theorem) is equal to the number of principal solutions corresponding to that root. It is to be regretted that no treatment is given for the case that the parameter takes a principal value.

The contents of the book before us have been described in some detail; what is to be said of it as a whole? That a physicist previously unacquainted with the properties of integral equations will succeed in obtaining any thorough familiarity with them from Kneser's presentation appears very doubtful. First a special problem, then a bit of theory, then more problems, and so on—theoretically this is an admirable plan for teaching or learning a subject; but in the present instance there is seldom a clear line of demarcation between what is always true, what is usually true, and what is true in some special case before us. To the novice the effect would probably be confusing in a high degree. Clearness is also not furthered by the author's building half the theory on the assumption that a symmetric kernel has at least one principal value, and then giving one special and two general proofs of this theorem at so late a stage that careful observation is needed to assure oneself that the vicious circle is avoided. It is the reviewer's belief that a more satisfactory order of presentation would have been obtained by considering first some one simple problem—for instance, one leading to the Fourier sine series; next proceeding to the general Fredholm theory, with the results of Schmidt and of Kneser himself for the symmetric kernel; and then taking up the many other special cases which are discussed.

But to one already familiar with the general theory of integral equations the book is of the highest value. Nowhere else are the details of the application to various physical problems so exhaustively discussed; nowhere else is seen so clearly the physical meaning, not merely of the broad outlines, but of the important separate notions in the theory. Kneser's work furnishes a mine of valuable material for illustrations which illuminate the true import of an integral equation.

That a text containing so many calculative manipulations as this does should have many misprints is to be expected. Some twenty-five have come to the reviewer's notice during a reading none too careful in examination of typographical details. Few are serious; some which might cause difficulty will be noted. On

page 78, line 20, for $-H$ read $-H/k(1)$. On page 87, line 12, each denominator Δ should be replaced by $k\Delta$. On page 97, line 8, for $1+|h'/\rho|$ read $2+|h'/\rho|$, and make corresponding changes in the succeeding lines. On page 98, line 2, for $-(P'+\epsilon_n)$ read $+(P'-\epsilon_n)$. On pages 190–197 there is continual confusion of the principal values and their reciprocals.

The general appearance of the page is clear and neat. The functional notation fx instead of $f(x)$ is not at present very widely used, but leads to no confusion here.

WALLIE ABRAHAM HURWITZ.

CORNELL UNIVERSITY,
January 14, 1913.

SHORTER NOTICES.

The Teaching of Mathematics in Secondary Schools. By ARTHUR SCHULZE. New York, The Macmillan Company, 1912. 16 mo. xx+367 pp.

IN these piping times when all readers of fifteen-cent magazines, and other patriots, are hastening to climb on the Progressive band wagon, there is grave offense in describing any person or thing as "conservative"; even the anæmic word "moderate" is eyed askance. We do not wish to create an unfavorable opinion of the book before us by attaching to it any of these unpopular predicates; we prefer to call it "eminently sane." The author is an experienced teacher, the difficulties that he faces are those that actually occur in practice, and the ways that he suggests to meet them are sensible and practical. Perhaps the book may be criticized for being a trifle too practical; a little more might be left to the imagination, there is a superabundance in the wealth of detailed illustration which becomes wearisome to the general reader. This is by design, not inadvertence, as the author shows in the preface (page vi) where, in referring to the books of Smith and Young he says: "This book covers a much more restricted field, but does it in greater detail." Perchance he is right. Surely there are a number of teachers who can obtain a good deal more benefit from a chapter on "The equality of triangles" with one hundred twenty-two illustrative examples, than from a comparison of the heuristic method with the individual mode.

The author opens by commenting with pleasing frankness on some of the present shortcomings of our schools; for instance (page 8):

"One would expect the schools to exert a wholesome influence in opposition to this ever growing shallowness. But far from it, they are the worst offenders. . . . There is more taught in many high schools during four years than the average human mind can assimilate in eight." Again (page 13):

"No other subject suffers so much and becomes so valueless as mathematics when treated by mechanical modes of study, and, on the other hand, no other secondary school subject is so admirably adapted to a judicious mode of study as mathematics."

These general considerations lead to the question of why mathematics should claim a place in the crowded secondary school curriculum. This is certainly a live question at present, and the author handles it in admirable fashion. He devotes both Chapters II and XVIII to a judicial balancing of the practical and the disciplinary in mathematical study. He thus replies to those objectors who, clothing themselves in a cloak of mystery called psychology (learnt in one course at summer school), maintain that there is no such thing as disciplining the human mind (page 26):

"If we should accept the theory that the general mental caliber of the student is not improved by study, it would undoubtedly be best to close all the schools after the fourth or fifth year of the grammar school, since the *knowledge* gained afterwards is not worth the trouble."

These considerations lead up to a discussion of the foundations of mathematics. The author takes the generally accepted view that modern researches into foundations have shown the utter futility of attempting to base school geometry upon a set of sufficient, categorical, and independent axioms. In like manner he is sceptical about spending much effort over the fundamental definitions (page 70):

"There exists no flawless definition of a straight line that is fit for school use, and undoubtedly the best policy would be to accept the term without definition."

"Explain an angle as a rotation by using a material contrivance that shows a rotation of a line . . . such illustration will show what an angle really is."

The preliminary chapters close with page 87 and the author

enters into a detailed study of the ever debatable subject of plane geometry. An idea of the topics discussed may be obtained from some of the chapter headings: The first propositions in geometry, Original exercises, Equality of triangles, Parallel lines, Limits, Regular polygons.

There is one merit of the author's treatment to which we must call particular attention, his insistence on the importance of a careful analysis of a geometrical problem before undertaking the constructive part of the proof. Here is an example (page 182).

"In equal circles the greater chords subtend the greater (minor) arc.

"*Query*: What is the only means we know to prove the inequality of arcs?

"*Answer*: Unequal central angles.

"*Q*. What therefore must we prove?

"*A*. $\angle O < \angle O'$.

"*Q*. What methods do we know for demonstrating the inequality of angles?" etc.

This scheme of question and answer, when printed at length, bears an unpleasant likeness to the catechism; it is, however, a vital part of geometrical teaching, and has received far too little attention in text-books great and small.

We have so far given the book much praise on didactic grounds; the same might well be continued to the end. Most unfortunately towards the middle the author begins to wabble in his mathematics, and since his work is written for teachers who have a right, if not to the whole truth, at least to nothing but the truth, we must pay some attention to this less attractive aspect of the work. The first difficulty arises in connection with the measurement of the angle between two lines which intersect within or without a circle. The author shows how, if we introduce the idea of positive and negative senses on the circumference, the two usual formulas may be reduced to one; he then continues (page 185):

"If we widen our definition by admitting imaginary arcs, the proposition is true even if one or both sides of the angle do not meet the circumference at all. Thus, if the vertex of the angle moves over the entire plane and its sides rotate in any manner, the proposition always remains true. It does not change abruptly at any point, but is continuous all over the plane. The principle applied here is often referred to as the principle of continuity."

It almost seems as if our author were consciously sinning against the light in writing this. What possible significance can a secondary school pupil attach to the words "imaginary arcs"? Can a teacher who refuses to define a straight line give his class any satisfactory notion of such things? What result can arise from such a process except to teach the pupil to pay himself with empty words? As for the principle of continuity, that has not even the primary merit of being always true. Take the theorem which scandalized the sophists of old: "The sum of two sides of a triangle is greater than the third side." We take an isosceles triangle ABC, where $AB = AC$; A, remaining always on the perpendicular bisector of BC, passes continuously into the imaginary domain and reaches such a position that the altitude $AH = \frac{1}{2}(BC)i$. Now the two equal sides have a length zero, the base is as before.

The author's next lapse occurs a few pages later. We are involved in a discussion of limits and the incommensurable case (page 191).

"We may either tell the student that the theorem can be proved for incommensurable numbers also, but that the proof is too difficult for school work, or we may attempt to make the incommensurable case more plausible by considering approximations of one of these numbers, for instance, the following approximation of $\sqrt{2} = 1.4,\ 1.41,\ 1.414,\ 1.4142$. Obviously the theorem is true for all approximations, hence the two numbers—the numerical measure of the angle and the numerical measure of the arc—can not differ by .1, .01, .001, .0001, etc. Or the error can not be as large as any number, however small, we may assign.

"We have thus proved there can be no *finite* difference between the numerical measure of angle and arc, and this is all the so called rigorous proofs with all their machinery accomplish."

It seems clear from this that the author has an uneasy notion that two constants which do not differ by any "finite" quantity may somehow differ by something else. Has he misunderstood the whole subject of infinitesimals in the calculus and carried away the idea that there are quantities which are less than any assignable quantity, but still not zero?

The discussion of plane geometry is so detailed that we are surprised to find solid geometry polished off in two short chapters. They are well written, especially the discussion of

the use of models and the principles for drawing geometrical solids, but many topics of first importance, as the measurement of curved surfaces, the treatment of triedral angles, etc., are omitted. It seems likely that, from this point on, the author felt himself cramped for room, thanks to his early prodigality; for whereas the introductory chapters and the plane geometry cover two hundred sixty-five pages, there are but one hundred pages left for all the rest of the mathematical curriculum. The introductory chapters in algebra are particularly good. The remarks on the choice of material, the placing of emphasis, and the teaching of factoring are excellent. We are less certain as to the didactic wisdom of his advice (page 329) to memorize the formula for solving a quadratic equation. It is far easier to memorize the formula than to understand what is really going on in the solution of a quadratic equation. Let the pupil do each equation at length until he has thoroughly mastered the what and the why, then, perhaps, let him memorize his formula to save time. It is possible also that the author is somewhat over enthusiastic in his praise of graphs; on page 333 we have seven separate reasons for their study, including "The study of graphs enables the student to solve many examples which otherwise he could not solve at all." Doth not the lady protest a little too much?

Unfortunately the algebraic part of the book is marred by mistakes related to those which occur in the geometry. On page 312 is a paragraph headed "The law of no exception." The suggestion of such a precious law at once challenges our interest: we read: "The scientific principle that guides us in such generalizations and that has been called the Law of No Exception or the Principle of Permanence of Equivalent Forms may be stated as follows. In the construction of arithmetic every combination of the previously defined operations ($+$, $-$, $\times$, etc.) shall be invested with a meaning, even when the original definition of the operation excludes such a combination; and the meaning imputed is to be such that the old laws of reckoning still hold good.'"

The credit for this profound statement is attributed to Schubert, and, in fact, we find it on page 14 of his Mathematical Essays and Recreations (Chicago, 1898). There is some obscurity clinging to the letters "etc.," but it seems fair to assume that they include the operation of division, in which case the principle reads: "Good news, we may divide by zero after all."

It is fair to say, that Schubert is entirely willing to take the responsibility for this interpretation, for we read four pages later in his work:

"We discover that, if we apply the ordinary rules of arithmetic to $a \div 0$, all such forms may be equated to one another, both when a is positive and when a is negative. We may, then, invent two new signs for such quotients $+\infty$ and $-\infty$."

We are not sure whether Schubert looks upon the use of this recumbent figure eight as a mathematical recreation. It certainly has no practical utility, it has no connection with the conception of a variable becoming infinite which is so fundamental in the calculus, and it does not come under any law of no exception since the old laws of reckoning do not all apply to it. But Schubert's book is not before us for review, and we prefer to assume that our present author copied this phrase inadvertently. Another inadvertence occurs on page 347:

"To invest $\sqrt{-1}$ and $\sqrt{-4}$ with a meaning, imaginary numbers must be introduced. . . . Imaginary numbers are just as real as other numbers." We do not wish to dispute this if the author will tell us what he means by an imaginary number; is it a real number-pair, a point in the Gauss plane, or merely a graphical symbol? There is no answer given to these questions; the most certain thing which we learn about an imaginary number is that it is real.

We seem to be closing this review with unfriendly comment; that is not the final impression which we wish to give. The faults of the book appear to us in the nature of "removable singularities," its merits are lasting.

J. L. COOLIDGE.

Anharmonic Coördinates. By Lieut.-Colonel HENRY W. L. HIME. Longmans, Green and Company. xiii+127 pp.

THE author's purpose in writing this book was to give a more detailed explanation of anharmonic coordinates than was given by their inventor, Sir W. R. Hamilton. Without laying any claim to originality, he has amplified Hamilton's outline to a degree that makes it quite ready reading as far as method is concerned, though there is a very noticeable amount of algebraic detail that is necessarily abbreviated. The first chapter is devoted to showing how a definite vector may be associated with any given point in the plane by means

of the Möbius net. The three numbers appearing in the expression for this vector and by which the point is fixed are called the anharmonic coordinates of the point. In chapter III, the equation of the straight line is developed from a condition on the coefficients of three coinitial vectors. Among the subjects treated in the other chapters are the general equation of the second degree, special conics, tangential equations, the anharmonic ratio, the involution, circles, and foci.

In regard to some details in the book, the reviewer would suggest omitting the words "of intersection" from line 7, section 8°, on page 14. Also it would seem better to use the parameters t and v homogeneously throughout section 9°, pages 14 and 15. In equation (16), page 17, read $\cos C(p_1q_2 + p_2q_1)$ instead of $\cos C(p_1q_1 + p_2q_1)$. The next form of this same equation displays without warning a change of notation that at first glance is rather puzzling. Half a line would state the change clearly. In line 5, page 20, read P_1' and P_2' for P_1 and P_2. In the line following equation (1), page 21, read $\Sigma^2 lx_2$ for Σlx_2. In the equation near the bottom of page 27, read $2(f\varphi_{x'} + g\varphi_{y'} + h\varphi_{z'})t$ for $2(f\varphi_{x'} + g\varphi_{y'} + h\varphi_{z'})$. In line 9, page 51, read X for IX. In line 10, page 54, read "the" for "some." At the bottom of page 63, read $p|pq_2r_3|$ and $p|pq_4r_3|$ for $r|pq_3r_2|$ and $r|pq_3r_4|$, the values of t and t' respectively. In line 2, section 5°, page 65, read D' for D. The value given for $C'D'$, page 67, is the reciprocal of the correct value. Likewise for the value of $B'C'$, and in addition read $|x_2y_3|$ for $|x_2y_2|$. The ditto marks on page 87 neglect the factor $a^2b^2c^2$. In the value for y'/z', page 88, read $|xy_1z_2|$ for $|xy_2z_2|$. In line 3 from the bottom of page 93, read c^2 for c^3.

These items suggest that the book is a little loosely put together in some respects; but it contains nevertheless much valuable material.

J. V. McKelvey.

Algebraische Kurven. Zweiter Teil: Theorie und Kurven dritter und vierter Ordnung. By Eugen Beutel. Sammlung Göschen No. 436. Leipzig, 1911. 16mo. 135 pp. Price, 80 Pf.

In a thin book of pocket size this treatise gives a large number of most precise definitions and theorems, fifty-seven well-executed cuts, and a variety of carefully worked out nu-

merical examples for illustration. Fifty pages are given to polars, the Hessian, duality, Plücker's formulas, and higher singularities. Curves of the third order fill thirty pages; curves of the fourth order, fifty; and there is an excellent index with nearly 150 references. Of course much is statement without proof, as in Pascal's Repertorium and the Encyklopädie; but a very considerable body of concise proof is included. The cuts are a specially admirable feature; many teachers who use lanterns in lectures will find them more available than those in Loria's collection.

Of the great mass of known theorems, for the most part only those are chosen which have direct bearing on the visible representation and the classification of curves; but this restriction permits relative fulness within limits. Concerning the related theory of invariants, elliptic and abelian functions, and covariant systems of curves, there is almost nothing here. But what is given is just what the beginner requires.

Extracts may be given to show that the author has his own point of view. "Zugleich ist ersichtlich, dass bei Anwendung von Linienkoordinaten die Aufgabe: die Umhüllungskurve einer nach einem bestimmten Gesetz sich bewegenden Geraden zu finden, rein algebraischer Natur ist; für Punktkoordinaten ist dies eine Aufgabe der Differentialrechnung" (page 31, § 3). Unless something is premised concerning the nature of the Gesetz and the terms in which it is expressed, this is a rash assertion. The statement intended was perhaps that a limit process is required in deducing the line equation of a curve from its point equation, or vice versa. On this problem the author gives a lucid discussion and (pages 38–41) very useful hints and examples.

As to ordinary and singular points and tangents (page 44) we find it stated that on point loci inflexional points and double tangents are ordinary features, while they are singularities on line loci; and dually for cusp tangent and double point. In spite of a plausible reason for this choice of words, it seems to the reviewer that the usual mode is better, namely to speak of inflexional *tangents*, with double tangents, as singularities on line loci, etc. For the one kind of *tangent* is, no less than the other, a part of the *projective* entity that we mean by point curve, and both alike are explicitly referred to in the line equation; we do not see how the substitution of the point of contact in place of its tangent can fail to confuse the student.

With these meticulous criticisms we may join a third. On page 67 we read: "Zwei Kurven derselben Klasse sind aber nicht immer ineinander projizierbar (vgl. z. B. die Dreiecks- und Viereckskurve dritter Ordnung S. 70). Die Einteilung nach der Klasse ist also keine projektive." A non sequitur is an agreeable rarity, and this has evidently, from the context, slipped in through some oversight.

In laying down this multum in parvo, we must commend the sections on quadratic transformation and the cuts exhibiting conics and their various related rational quartics.

H. S. WHITE.

Lehrbuch der Mathematik für Studierende der Naturwissenschaften und der Technik. Einführung in die Differential- und Integralrechnung und in die analytische Geometrie. Von Dr. GEORG SCHEFFERS. Zweite verbesserte Auflage. Leipzig, Veit, 1911. 8vo. vi+732 pp.

IMAGINE a course of some 150 lectures on algebra, trigonometry, analytic geometry, and calculus given by a sound mathematician and an excellent teacher, not lacking in the sense of humor. Imagine the audience to be students of general science or engineering who have taken the usual secondary school courses in mathematics, "jedoch manches davon wieder vergessen, vielleicht auch manches davon nicht ganz verstanden haben." (From the preface.) Imagine these lectures together with all side remarks, illustrations and blackboard drawings and sketches taken down word for word by a good stenographer, whose notes are transcribed and published in a large octavo volume by a first rank Leipzig firm. Imagine all this and from one point of view the reader will have a good idea of the book under review.

The word "function" dominates the plan of the work. If we call our usual division of college mathematics into algebra, trigonometry, analytic geometry, and calculus a horizontal division, we might call Scheffers' division a vertical one. Beginning with the notion of a function, he takes up one after the other, linear, quadratic, rational integral, rational, logarithmic, exponential, and trigonometric functions. An outline of his chapter on the quadratic function will give an idea of his method of treatment. The graph of the function is discussed in great detail, beginning with the simple case x^2 and then taking up more complicated cases with numerical

coefficients, gradually leading up to a discussion of the graph of $ax^2 + bx + c$ and the changes due to letting one or more of the coefficients vary. Then the idea of "slope" at a point, which had been previously discussed in connection with the linear function, is taken up and leads to a general discussion of limits, and in particular to the derivative of the quadratic function.

In a general way the chapters on the other functions follow a similar plan and integration is gradually introduced. At the end of the chapter on the rational function a chapter, "Einiges aus der analytischen Geometrie," rounds out the course in analytic geometry which has been running through the previous chapters, though polar coordinates are not discussed until the graph of the logarithmic spiral is taken up. The chapter on trigonometric functions, while lacking the solution of triangles, is more complete than our usual freshman course.

After the chapter on trigonometry, which finishes the first half of the volume, the author swings into a regulation course in calculus, and from this point on the matter is given more in the usual text-book manner, including the topics generally given in connection with functions of one and of more than one variable. However the genial conversational style is used to the end of the last chapter.

This book is one of the best examples of a class of books common in Germany but rare in this country, i. e., a book written especially to be used without a teacher. The author in his preface refers to it as, "Lehrbuch für Anfänger und solche, die es bleiben wollen." It is a mine of interesting problems, which if not all applied problems are at least clothed in the language of applications. At some points in the early chapters the weight of detail is so marked that the author takes it upon himself to advise the better class of students to skip a few pages and provides rather complete summaries and "Rückblicke" for their use. At the end of the book a few tables are given, among them a table of integrals, and one of hyperbolic functions. The work will be very interesting to those teachers who are interested in the problem of combining our freshman and sophomore courses into one harmonious "course in mathematics."

A. R. CRATHORNE.

Notions de Mathématiques. Par A. SAINTE-LAGUË. Avec Préface de G. KOENIGS. Paris, A. Hermann et Fils, 1913. vii + 512 pp. Price 7 francs.

In France, during the past decade, the baccalauréat de l'enseignement secondaire has been granted to students who have successfully completed the seven year course of the lycée, in one of four main lines of study. In the seventh year these classes of students are characterized as of Philosophie *A*, Philosophie *B*, Mathématiques *A*, or of Mathématiques *B*. All students of the first two classes have studied both Latin and Greek, in the third class Latin and modern languages but no Greek; in the fourth class, no students have had Greek, few have taken up Latin but all have had broad training in modern languages. Prior to the seventh year those in the Philosophie group have devoted 10.5 to 11 per cent. of all their recitation periods to mathematics; those in the Mathématiques group 19.4 to 22.8 per cent. From the latter group come the future mathematicians.

It was with the needs of the students of the classes Philosophie *A*, *B* in mind that M. Jules Tannery wrote his most interesting Notions de Mathématiques* to which are appended 25 pages of Notions Historiques by his brother Paul. Although Tannery's work is largely in conformity with the *programme*, the whole reads as a freshly told story. About a third of the book is devoted to an "Introduction." With particular insistence on the accurate definition of all terms used, the following subjects are treated in nine chapters: identities; algebraic geometry; equations of the second degree; coordinates; empirical curves; notions of analytical geometry (40 pages); tangents, velocity derivatives; notions of the integral calculus; limits; infinitesimals, definite integrals, series. The student cannot fail to be interested by the way in which the various subjects are welded into a homogeneous whole.

M. A. Sainte-Laguë, "professeur de mathématiques spéciales" in the lycée at Besançon and an Ecole Normalian of 1903, has followed in the steps of his former master by now publishing a book with the same title as the one to which we have just referred. But though it is much larger, the topics

* A German edition has been published (1909) with the title Elemente der Mathematik. Cf. the BULLETIN, April, 1911, vol. 17, pp. 367–368. About 20 pages of "Notions d'Astronomie" are appended to the French editions since 1905.

treated are fewer in number and the whole method of discussion is radically different. At most universities of France a course in mathématiques générales is offered for students of physics, chemistry and engineering. Algebra, analytical geometry, analysis, and mechanics are here developed.* Largely as a preparation for such courses and to fill up lacunæ in connection with them, M. Sainte-Laguë's book was written. While rigor of presentation is not neglected, details in proofs are not always dwelt upon and practical applications of the various subjects are emphasized.

To contrast with Tannery, the first section (pages 1–81) treats of arithmetic; the next section (pages 82–202) of algebra, including derivatives; plane trigonometry, pages 203–234; under geometry (pages 235–399) the sub-headings are: lines and planes, parallels, spherical geometry, metrical relations, lengths, areas and volumes ("formule de Tchebitcheff" is used on page 347 and page 502 but this spelling is not sanctioned by either Cantor or *Bibliotheca Mathematica*), graphic constructions, descriptive geometry, methods in geometry; kinematics, pages 399–416.

At the end of every section are references to 500 exercises for solution (pages 417–470). These are mostly numerical and letters *A, B, C* indicate the degree of their difficulty. Then follow various numerical tables including one of logarithms (four place), formulas, etc. The whole concludes with an admirable "index alphabétique" as well as "table des matières" (pages 503–512).

Anyone somewhat familiar with the French educational system will find this volume of interest. In connection with both the theory and the problems there is suggestive material for early undergraduate college teaching.

R. C. ARCHIBALD.

Encyklopädie der Elementar-Mathematik. Angewandte Elementar-Mathematik. Zweiter Teil. Dritter Band. Zweite Auflage. Von HEINRICH WEBER und JOSEF WELLSTEIN. Leipzig und Berlin, Teubner, 1912. xiv+671 pp. 14 Marks.

* A representative treatment of the subject is given in E. Fabry's Traité de Mathématiques générales, 2e éd., Paris, 1911, and the key to the problems, Problèmes et Exercices de Mathématiques générales, Paris, 1910. The first editions of these books have been reviewed in the BULLETIN, vol. 15 (1909), pp. 395–399 and vol. 17 (1911), p. 320.

This second edition of the Elementary Encyclopedia has received such extensive additions that the third volume of the original appears in two parts. The first of these was reviewed in this BULLETIN, page 87 of the current volume. The second part, under consideration, contains the revised books entitled "Graphik" and "Wahrscheinlichkeitsrechnung." The first book has a new section on "Axonometrie und Perspektive." "Two new books have been added to meet the views of certain critics of the first edition: "Politische Arithmetik" and "Astronomie." Other changes are minor.

The third book includes the theory of interest and actuarial computations. The theory of interest is based upon compound interest, in the sense that simple interest is looked upon as an annuity in perpetuity. Only the elements of insurance are developed.

The fourth book deals with spherical astronomy and the calculation of orbits. The subjects considered are astronomical coordinates, determination of time, variations of stellar coordinates, observations with instruments, determination of latitude and longitude, and orbits.

The additions to this useful work will be welcome in many quarters. While one might criticize the proportional amount of space devoted to them, and to the other divisions of the book, such criticism would arise from purely personal views as to what applications are important, and would vary from person to person. The authors and editors are deserving of praise for the work taken as a whole.

JAMES BYRNIE SHAW.

A History of the Theories of Aether and Electricity from the Age of Descartes to the Close of the Nineteenth Century. By E. T. WHITTAKER. London, Longmans, Green, and Co., 1910. xiii+475 pp.

EITHER consciously or unconsciously, Whittaker must be imbued with a missionary spirit which leads him forth into dark places to enlighten them with opportune gospel. Three of his books, Modern Analysis (1902),* Analytical Dynamics (1904),† and this History, bear ample evidence to this.

We do not lack for works on the theory of functions, but

* Reviewed in the BULLETIN, volume 10, p. 351, by M. Bôcher.
† Reviewed in the BULLETIN, volume 12, p. 451, by E. B. Wilson.

they are unfortunately similar and similarly placed. Take for example Osgood's Funktionentheorie, a consummate work of art, delightfully fit the student of pure mathematics, but so completely concerned with method and point of view and beauty that the student who must use his analysis gets little from it. Indeed most modern works on analysis are modern to the point of abolishing analysis. The prime and unique feature of Whittaker's Modern Analysis is the welding together of modern method with the older analytic facility so that the whole may be of use to the physicist and astronomer.*

We do not lack for works on mechanics; but Whittaker's is unique, and again the uniqueness consists in the amalgamation of the old with the new, of the admirable English problem-solving with the theoretical advances and advantages of integral invariants, continuous groups, and the like.

It is obvious that this power to judge values, to pick and combine the essentials in different points of view, is vitally necessary to the successful composition of a history of ether and electricity such as is here offered to the public. It is fortunate that one who has shown the power so clearly should have undertaken the work and brought it forth at a time particularly opportune.

There is no time at which a well-coordinated history of a vast branch of science can be considered inopportune, but the years when a great theory has at last conquered the world after considerable opposition and is taught far and wide by that conservative element who, had they been alive and teaching during its incipiency, would have ignored it or fought it, the years when the progressive element are looking forward to new points of view, to new theories, not yet thoroughly formulated,—these years are indeed the best in which such a history may appear.

Relative to the ether we are now in precisely this sort of period. Maxwell's theory of action through an all-pervading plenum has had its triumphal acceptance. Those who could not or would not understand the theory have for the most part passed away. One of the greatest and one of the last of them was Lord Kelvin. He was a deep student of fluid and of elastic media, he was ever seeking an intelligible mechanical

* We do not wish at all to impugn any of Bôcher's criticisms in the review just cited as to the incompleteness of the rigorous treatment in some parts of Whittaker's Modern Analysis.

conception of the ether; he apparently never found one which was completely satisfactory to him, and it is doubtful if he ever became a real sympathizer with Maxwell's ether. To all this the publication of his Baltimore lectures in 1904 bears witness.

These difficulties which bothered Kelvin and which troubled everybody in the early days of the theory have by no means all been resolved; they have merely been ignored. The real triumph has not been physical but psychological; we no longer ask those awkward questions which are inimical to the theory, we take the whole fabric as we find it and unquestioningly make application of it. If there be questions, they are of a different sort.

In recent years some active minds have been looking forward toward the formulation of new theories, toward the abolition of the ether. The theory of relativity and the hypothesis of energy quanta have been the two ideas upon which they have chiefly focused their attention. If they in their turn shall triumph, it will probably be not for the reason that all the questions which the opponents of the theories now bring forth shall have been satisfactorily answered, but because the questioners shall have ceased to question. We advance by ignoring our known ignorance and by concentrating upon our assumed knowledge.

We are living at a time of (at least attempted) transition, and that is the opportune time for Whittaker's History to appear. The author himself with his true insight and admirable balance seems to recognize this, and to state it well in the closing paragraphs of the work.

Chapter I contains an account of the theory of the ether in the seventeenth century, founded upon the rather vague speculations of Descartes, but very influential owing to the sway of Descartes over the minds of scientists for a considerable time. Light was the chief physical phenomenon which at this time was subject to experiments sufficiently accurate to test a theory, and Newton and Huygens are the chief names. In Chapter II we turn our attention to electric and magnetic science prior to the introduction of the potentials. Here we are in the domain of action at a distance. Chapter III is on galvanism from Galvani to Ohm.

With Chapter IV we come to the luminiferous medium from Bradley to Fresnel, though during some of the period the idea

of a medium was not very strong. The ether as an elastic solid is the subject of Chapter V. Here we are still interested in a luminiferous medium and, with the exception of Boussinesq's work and some of W. Thomson's, we are dealing with the writers of the first half of the last century. It is of course impossible for the author to arrange everything in rigorously chronological order; that would violate too greatly the logical sequence. The analysis of various optical theories, other than the electromagnetic, as set forth in these two chapters will be highly useful to teachers of optics.

The work of Faraday is the almost exclusive topic in Chapter VI. This would naturally lead to the work of Maxwell (Chapter VIII) without interruption were it not for the fact that the mathematical electricians of the middle of the nineteenth century, whose work could hardly have been done before Faraday's experiments, adopted as the basis of their work the conception of action at a distance instead of Faraday's physical conceptions of lines of force. The work of these mathematicians is therefore analyzed in Chapter VII between Faraday and Maxwell.

Chapter IX discusses models of the ether (subsequent to Maxwell). The contributions of the more immediate followers of Maxwell are taken up in Chapter X. Under the title of conduction in solutions and gases from Faraday to J. J. Thomson we return in Chapter XI to discrete theories of electricity; and in the concluding Chapter XII is found an account of the theory of the ether and electrons in the closing years of the nineteenth century, at the very close of which comes Richardson's work on thermionics, belonging actually to the twentieth century.

One theory which might properly have been mentioned, but was not, is that contained in Reynolds's Submechanics of the Universe. Here is a discrete ether and an exceedingly complicated mathematical investigation, which seems both worthy and needful of explanation to the readers of this History. With the exception of this omission from Chapter IX, we find no point for adverse criticism.

To go into further detail with regard to the contents of this History, which should and will be widely read, is needless. Suffice it to say that a careful study of all of the work twice, and of many portions of it several times, leaves but one resolution, namely, to continue the study indefinitely; for there

is always something new to learn where so much material is so well presented. EDWIN BIDWELL WILSON.

Theorie der elliptischen Funktionen. Von M. KRAUSE unter Mitwirkung von E. NAETSCH. Leipzig, B. G. Teubner, 1912. vi+186 pp.

ANOTHER text in Jahnke's series for engineers and students. Its object is to give a brief development of elliptic functions for the sake of rendering intelligible those formulas, figures, and tables which relate to elliptic functions in Jahnke and Emde's Funktionentafeln. The titles of the chapters are: Introduction, General theory of Jacobi's functions, Special theory for the real domain, Legendre's normal integrals, Weierstrass's functions, Representation of the general doubly periodic function by means of the foregoing types, Reduction of the general elliptic integral to normal forms. The development is based on the ϑ-functions, and makes relatively little use of the theory of functions of a complex variable. The prominent place given to the ϑ-functions is commendable. In most cases these series converge with extraordinary rapidity and are readily available for computation. The attention to the functions *sn*, *cn*, *dn* is also advantageous; in physical problems where the trigonometric functions offer a first approximation, these elliptic functions are the most natural to use. The p-function is admirably discussed, and especial mention should be made of the reduction of the p-function with conjugate imaginary periods to the related p-function with real and pure imaginary periods. It is noteworthy that the authors use a plain p, and not $\wp$; perhaps this latter contortion is on the road to abandonment.

From some points of view it might have been better to assume and use a greater, even a great, amount of the theory of functions of a complex variable; the work would not have been so elementary, but it would have been more instructive. We note with regret that Jahnke has not announced in his series a text on the theory of functions. Such a text, properly executed in the interest of physicists and engineers, would be a welcome addition to his series. Perhaps Lewent's Konforme Abbildung will supply much of the lack; for it is in connection with conformal representation (and elliptic functions) that the function theory becomes most vital to the student of applied mathematics. Whether such a student will

get as much out of Krause's work as he would out of one constructed more along the lines of Greenhill's Elliptic Functions is a matter of considerable doubt, but at any rate we have a neat and reasonably short exposition which admirably serves its announced purpose of orienting the reader in the corresponding part of Jahnke's tables. E. B. WILSON.

An Introduction to Thermodynamics. By JOHN MILLS. Boston, Ginn and Co., 1910. viii+136 pp.

The brief text on thermodynamics by Mills shows that the author has read and digested a large number and a large variety of works on the subject, and that he knows how to select from this diversity the elements he needs and combine them into a carefully coordinated sequence which shall serve to lead the pupil from his elementary work on heat through so much of thermodynamics as may be necessary for the ordinary student of engineering. The simpler notions and notations of the calculus are constantly used; a large number of numerical practical problems are worked in the text, and a set of miscellaneous exercises for the reader is furnished at the end. The style is concise, but clear, and the various physical concepts are defined with the accuracy of the physicist rather than with the frequent inaccuracy of the engineer. The titles of the chapters are: Fundamental concepts and laws, Gases, Water and its saturated vapor, Superheated steam, Flow of steam and gases. The page has that attractive appearance which generally goes with the imprint of the Athenæum Press. E. B. WILSON.

Annuaire du Bureau des Longitudes pour l'An 1913. Paris, Gauthier-Villars. 16mo.

THE editors of the Annuaire have clearly decided that it should be kept fully up to date. Several of the tables of constants are again improved, some by a recasting of the contents, others by the addition of new matter, and still others by the adoption of the latest and best values obtainable. Any one interested in its use will find these changes briefly but clearly set forth in the preface. The information is easy to find with the help of the full index. The main defect is a minor one and perhaps a matter of opinion: the edges are uncut and there are some 800 pages.

The Notices contain, besides the speeches made at the

obsequies of Radau and Poincaré, who had both assisted for some years in the preparation of the Annuaire, an article on the application of wireless telegraphy to the distribution of the daily time by Commandant Ferrié, and a resumé by M. Bigourdan of the observations made during the solar eclipse of 1912, April 17. From these we learn that the line of totality was almost exactly midway between those predicted by the Connaissance des Temps and the American Ephemeris.

ERNEST W. BROWN.

NOTES.

THE twentieth summer meeting and seventh colloquium of the American Mathematical Society will be held at the University of Wisconsin, Madison, Wis., during the week beginning Monday, September 8, 1913. The first two days will be devoted to the regular sessions for the presentation of papers. The colloquium will open on Wednesday morning and will close on Saturday morning. Courses of lectures will be given as follows (the list of principal topics is appended):

PROFESSOR L. E. DICKSON: "Certain aspects of a general theory of invariants, with special consideration of modular invariants and modular geometry."

A function-theoretic basis for a general theory of invariants applicable to both algebraic and modular invariants; concrete examples.

Geometrical derivation of a fundamental system of invariants of a binary modular group; application to the invariantive classification of binary forms.

The so-called form problem for a modular group; solution in the simple, but typical, case of two variables. Finiteness of modular covariants; examples of fundamental systems.

General modular geometry; the projective geometry and covariant theory of a conic and of a quadric surface modulo 2; certain features of the modular geometry of cubic and quartic curves and surfaces.

PROFESSOR W. F. OSGOOD: "Topics in the theory of analytic functions of several complex variables."

The lectures will attempt to give a brief survey of what

has been accomplished in the study of some of the more important problems of this branch of analysis. The topics will be drawn mainly from the following, and the lectures will cover as wide a range as is practicable.

The problems which were first studied: the problem of inversion in the theory of the Abelian integrals; periodic functions and the Riemann-Weierstrass theta theorem; modular functions of several variables (Hilbert, Blumenthal).

The simplest singularities, and allied theorems relating to analytic continuation; Weierstrass, Cousin, Hahn, Hartogs. Weierstrass's condition in the space of analysis that a function be rational, or algebraic.

Residues of multiple integrals, and algebraic functions of two variables; Poincaré, Picard, Simart, Hensel.

Homogeneous variables and the Schottky-Klein prime function.

THE next meeting of the British association for the advancement of science will be held at Birmingham, September 10–17, under the presidency of Sir OLIVER LODGE. Dr. H. F. BAKER is chairman of Section A (mathematics and physics).

AT the meeting of the London mathematical society, held on March 13, the following papers were read: By J. PROUDMAN, "Some cases of tidal motion of rotating sheets of water"; by L. J. MORDELL, "Indeterminate equations of the third and fourth degree."

THE Paris academy of sciences announces the following problem for the subject of the Bordin prize, to be awarded in 1915:

"To make an important advance in the theory of curves of constant torsion; to determine, if possible, under what conditions such curves are algebraic, or at least when unicursal."

THE Swiss mathematical society held a special meeting March 9, 1913, at Neuchatel, to discuss the question of mathematical instruction in the Swiss universities, on the basis of the report of the Swiss sub-committee of the international commission. The next regular session will be held September 9, at Frauenfeld, under the presidency of Professor H. FEHR.

THE following doctorates in mathematics were conferred by the University of Paris in 1912. The subject of the thesis is appended.

P. HELBRONNER: "Résumé des opérations exécutées jusqu'à la fin de 1911 pour la description géométrique détaillée des Alpes françaises."

P. LEVY: "Sur les équations intégro-differentielles définissant des fonctions de lignes."

E. TURRIÈRE: "Sur les congruences des normales qui appartiennent à un complexe donné.

J. BOSLER: "Sur les relations des orages magnétiques et des phénomènes solaires."

H. GALBBUN: "Sur la représentation des solutions d'une équation linéaire aux différences finies pour les grandes valeurs de la variable."

C. NICOLAU: "Sur la variation dans le mouvement de la lune."

COLUMBIA UNIVERSITY. The following advanced courses in mathematics are announced for the summer session, July 7 to August 15. All courses are five hours a week. By Professor C. J. KEYSER: Modern theories in geometry; History and significance of central mathematical concepts.—By Professor JAMES MACLAY: Higher algebra; Elliptic functions. —By Professor EDWARD KASNER: Continuous groups.—By Professor W. B. FITE: Theory of functions of a real variable.

THE following courses in mathematics are announced for the academic year 1913–1914.

COLUMBIA UNIVERSITY. By Professor C. J. KEYSER: Modern theories in geometry, three hours; History and significance of central mathematical concepts, three hours.—By Professor T. S. FISKE: Differential equations, three hours, first half-year; Theory of functions of a real variable, three hours.—By Professor F. N. COLE: Theory of functions of a complex variable, three hours; Theory of groups, three hours. —By Professor JAMES MACLAY: Theory of numbers, three hours; Elliptic functions, three hours.—By Professor D. E. SMITH: History of mathematics, three hours.—By Professor

EDWARD KASNER: Seminar in differential geometry, three hours, first half-year.—By Professor W. B. FITE: Infinite series, three hours, second half-year.—By Professor H. E. HAWKES: Higher algebra, three hours, first half-year.—By Dr. H. W. REDDICK: Differential equations, three hours, second half-year.—By Dr. N. J. LENNES: Theory of point sets, three hours.

CORNELL UNIVERSITY.—By Professor J. MCMAHON: Fourier series and spherical harmonics, three hours; Insurance and probabilities, two hours.—By Professor J. I. HUTCHINSON: Elliptic functions, two hours.—By Professor V. SNYDER: Geometry on an algebraic surface, two hours.—By Professor F. R. SHARPE: Differential equations, two hours; Vector analysis, three hours.—By Professor W. B. CARVER: Projective geometry, three hours.—By Professor D. C. GILLESPIE: Advanced calculus, three hours.—By Dr. C. F. CRAIG: Theory of linear differential equations, three hours.—By Dr. F. W. OWENS: Foundations of geometry, three hours.—By Dr. J. V. MCKELVEY: Advanced analytic geometry, three hours.—By Dr. L. L. SILVERMAN: Theory of numbers, three hours (second term).—By Dr. W. A. HURWITZ: Theory of finite groups, three hours (first term); Algebraic equations, three hours (second term). The mathematical club will meet every Monday.

HARVARD UNIVERSITY.—By Professor B. O. PEIRCE: Potential function, two hours (first half-year).—By Professor W. F. OSGOOD: Advanced calculus, three hours; Dynamics, second course, three hours; Theory of functions, second course, three hours (second half-year); Theory of functions, first course, three hours, with Professor BÔCHER.—By Professor M. BÔCHER: Fourier's series, Bessel's and Legendre's functions, three hours (second half-year).—By Professor C. L. BOUTON: Differential equations, with Lie's theory, three hours; Introduction to modern geometry and modern algebra, three hours, with Mr. GRAUSTEIN.—By Professor J. L. COOLIDGE: Probability, three hours; Algebraic plane curves, three hours.—By Professor G. D. BIRKHOFF: Infinite series and products, three hours (first half-year); Problem of three bodies, three hours.—By Dr. D. JACKSON: Distribution of primes, three hours (second half-year).—By Dr. F. J. DOHMEN: History of mathematics, three hours (first half-year).—By

Mr. W. C. GRAUSTEIN: Advanced algebra, three hours (first half-year); Differential geometry, three hours (second half-year).

Various courses in reading and research are also offered on special topics, and Professor Birkhoff and Dr. Jackson will conduct a fortnightly seminar in analysis.

THE following courses in mathematics are announced for the present semester:

UNIVERSITY OF PARIS.—By Professor P. APPELL; Analytic mechanics, two hours.—By Professor E. PICARD: Analytic functions and integral equations, two hours.—By Professor E. GOURSAT: Differential equations, two hours.—By Professor C. GUICHARD: General laws of motion, two hours.—By Dr. VESSIOT: Elements of analysis and of mechanics, two hours.—By Professor H. ANDOYER; Theoretic astronomy, two hours.—By Professor J. BOUSSINESQ: Theory of waves, two hours.—By Professor G. KOENIGS: Theory of thermic motors, two hours.—By Dr. L. CAHEN: Theorem of Fermat, two hours. Conferences.—By Dr. L. LEBESGUE: Geometric applications of integral calculus, two hours.—By Professor J. DRACH: Rational mechanics, two hours.—By Professor H. ANDOYER; astronomy, one hour.—By Dr. L. SERVANT: Graphical statics, one hour.

PROFESSOR F. ENGEL, of the University of Greifswald, has accepted a call to the University of Giessen, as successor to Professor E. NETTO.

DR. H. JUNG, of Hamburg, has been appointed professor of mathematics in the University of Kiel.

DR. H. CHATELET has been appointed associate professor of mechanics at the University of Toulouse.

DR. C. GUICHARD has been appointed professor of general mathematics at the University of Paris.

DR. H. MOHRMANN, of the technical school at Carlsruhe, has accepted the professorship of mathematics at the mining academy of Clausthal.

DR. TH. PÖSCHL, of the technical school at Graz, has been appointed associate professor of mathematics at the German technical school of Prague.

PROFESSOR E. HELLEBRAND has been appointed professor of mathematics at the agricultural institute of Vienna.

DR. T. v. KÁRMÁN, of the University of Göttingen, has been appointed professor of mechanics and aerodynamics at the technical school at Aachen.

PROFESSOR G. MAJCEN, of the University of Agram, has been elected corresponding member of the Bohemian academy of sciences.

DR. A. SIGNORINI has been appointed docent of rational mechanics at the University of Padua.

PROFESSOR A. R. FORSYTH, formerly of Cambridge University, has accepted the professorship of mathematics at the Imperial College of science and technology, London.

DR. H. B. HEYWOOD has been appointed assistant lecturer in mathematics at Bedford College for ladies at London.

MISS H. P. HUDSON, of Newnham College, Cambridge, has accepted the professorship of mathematics at the West Ham technical school.

THE following instructors in mathematics have been appointed at Harvard University for the academic year 1913–1914: F. J. DOHMEN, H. D. GAYLORD, J. S. MIKESH, W. E. MILNE, R. B. ROBBINS, C. E. WILDER, L. T. WILSON.

DR. T. H. GRONWALL has been appointed instructor in mathematics at Princeton University.

PROFESSOR P. F. SMITH, of Yale University, has been granted leave of absence during the first half of the academic year of 1913–1914.

PROFESSOR J. C. FIELDS, of the University of Toronto, has been elected to membership in the Royal society of London.

PROFESSOR R. W. PRENTISS, since 1891 head of the department of mathematics and astronomy at Rutgers College, died on April 5 at the age of fifty-six years. Professor Prentiss had been a member of the American Mathematical Society since 1892.

Professor MARIO PIERI, of the University of Parma, died February 28, at the age of 53 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALTIN (R.). See MÜLLER (H.).

BARBETTE (E.). Les carrés magiques du m^{ième} ordre. Liège, Pholien, 1912. 8vo. 244 pp. Fr. 7.50

——. Les piles merveilleuses. Liège, Pholien, 1912. 8vo. 16 pp.

BERNAYS (P.). Ueber die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nichtquadratischen Diskriminante. (Diss.) Göttingen, 1912.

BIANCHI (L.). Lezioni sulla teoria aritmetica delle forme quadratiche binarie e ternarie. Pisa, 1912. 8vo. 11+701 pp. Lithographed.

BÔCHER (M.). Introduction to higher algebra. Prepared with the cooperation of E. P. R. Duval. New reprint. New York, Macmillan, 1912. 8vo. 315 pp. Cloth. $1.90

BOIS-REYMOND (P. DU). Zwei Abhandlungen über unendliche (1871) und trigonometrische (1874) Reihen. Herausgegeben von P. E. B. Jourdain. 1te und 2te Abhandlung. (Ostwald's Klassiker der exakten Wissenschaften. Nr. 185.) Leipzig, Engelmann, 1912. 8vo. 115 pp. M. 3.75

——. Abhandlung über die Darstellung der Funktionen durch trigonometrische Reihen (1876). Herausgegeben von P. E. B. Jourdain. 3te Abhandlung. (Ostwald's Klassiker. Nr. 186.) Leipzig, Engelmann, 1912. 8vo. 140 pp. M. 5.00

BOUFFALL (S.-A.-F. DE). Démonstration complète du grand théorème de P. Fermat. Varsovie, 1912. 8vo. 5 pp.

——. Deuxième démonstration complète du grand théorème de Fermat. Varsovie, 1912. 8vo. 8 pp.

BRANDENBURG (H.). Der grosse Fermatsche Satz und sein Beweis. 2te Ausgabe. Leipzig, Lorentz, 1913. 8vo. 8 pp. M. 0.60

CAJORI (F.). An introduction to the modern theory of equations. New reprint. New York, Macmillan, 1912. 8vo. 239 pp. Cloth. $1.75

CAMPBELL (D. F.). The elements of the differential and integral calculus. New reprint. New York, Macmillan, 1912. 12mo. 362 pp. Cloth. $1.90

CLEMENT-JONES (A.). Introduction to algebraical geometry. London, 1912. 8vo. 548 pp Cloth. 12s. 6d.

ELTE (E. L.). The semiregular polytopes of the hyperspaces. Groningen, 1912. 8vo. 144 pp.

ENESTRÖM (G.). Verzeichnis der Schriften Leonhard Eulers. 2te Lieferung. (Jahresbericht der deutschen Mathematikervereinigung. Ergänzungsband.) Leipzig, Teubner, 1913. 8vo. Pp. 209–388. M. 10.00

FIELDS (J. C.). On the foundations of the theory of algebraic functions of one variable. (Royal society.) London, Dulau, 1913. 4to. 36 pp. 1s. 6d.

GAEDE (E.). Ueber den Anteil der Logik, Methodologie und Erkenntnistheorie an den theoretischen Wissenschaften. (Diss.) Erlangen, 1912.

HÄFNER (P.). Einführung in die Differential- und Integralrechnung für höhere Techniker. Stuttgart, 1912. M. 16.00

HATHAWAY (A. S.). A primer of quaternions. 2d edition. New York, Macmillan, 1911. 12mo. 113 pp. Cloth. $0.90

HINTIKKA (E. A.). Ueber das Verhalten der Abbildungsfunktion auf dem Rande des Bereiches in der konformen Abbildung. (Diss.) Helsingfors, Finnische Literatur-Gesellschaft, 1912. 4to. 8+36 pp.

JANS (C. DE). Les multiplicatrices de Clairaut. Contribution à la théorie d'une famille de courbes planes. Gand, Hoste, 1912. 8vo. 4+136 pp. Broché. Fr. 5.00

JOURDAIN (P. E. B.). See BOIS-REYMOND (P. DU).

LÖWENKLAU (L.). Zum grossen Fermatschen Satz. 2ter und endgiltiger Beweis. Dresden, Köhler, 1913. 8vo. 4 pp. M. 0.50

MEHMKE (R.). Vorlesungen über Punkt- und Vektorenrechnung. 1ter Band: Punktrechnung. 1ter Teilband. (Teubner's Sammlung. Nr. 37.) Leipzig, Teubner, 1913. 8vo. 8+394 pp. M. 14.00

MIKAMI (Y.). The development of mathematics in China and Japan. (Abhandlungen zur Geschichte der mathematischen Wissenschaften. Heft XXX.) Leipzig, Teubner, 1913. 8vo. 350 pp. M. 19.00

MÜLLER (E.). Das Abbildungsprinzip. (Antrittsrede.) Wien, Verlag der technischen Hochschule, 1912. 8vo. 29 pp.

MÜLLER (H.) und BALTIN (R.). Graphische Darstellungen. Graphische Behandlung von Gleichungen. Grundlehren von den Kegelschnitten. Leipzig, Teubner, 1912. M. 11.00

NEUMANN (E. R.). Beiträge zu einzelnen Fragen der höheren Potentialtheorie. (Preisschrift.) Leipzig, Teubner, 1912. 8vo. 23+188 pp. M. 11.00

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RATSCHLÄGE für die Studierenden der Mathematik und Physik an der Universität Jena. 3te Auflage. Jena, 1912.

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SCHACHT (W.). Beweis des grossen Fermatschen Satzes. Lausanne, Frankfurter, 1912. 8vo. 16 pp. M. 2.00

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TOULOUSE (E.). Henri Poincaré; enquête médico-psychologique sur la supériorité intellectuelle. Paris, 1912. 8vo.

VOLTERRA (V.). Leçons sur l'intégration des équations différentielles aux dérivées partielles. Nouveau tirage. Paris, Hermann, 1912. 4to. 3+4+83 pp.

——. Trois leçons sur quelques progrès récents de la physique mathématique. (Lectures delivered at the celebration of the twentieth anniversary of the foundation of Clark University.) Worcester, Mass., Clark University, 1912. 8vo. 82 pp.

WACKER (H.). Ueber Differentialgleichungen vom Fuchsschen Typus mit einem Parameter und ihre Reduzibilität. (Diss.) Strassburg, 1912.

II. ELEMENTARY MATHEMATICS.

FINE (H. B.) and THOMPSON (H. D.). Co-ordinate geometry. New reprint. New York, Macmillan, 1912. 12mo. Half leather. $1.60

FUJISAWA (R.). Summary report on the teaching of mathematics in Japan. Tokio, 1912.

GAJDECZKA (J.). Auflösungen von arithmetischen und geometrischen Textaufgaben. Wien, 1912. 8vo. 5+165 pp. M. 2.50

GALDEANO (Z. G. DE). Ensayos de sintesis matemática y nuevo metodo de enseñanza matemática. Parte primera. Zaragoza, Casañel, 1911. 8vo.

——. Nuevo método de enseñanza matemática. Zaragoza, Casañel, 1912. 8vo. 24+56 pp.

GOLDZIHER (C.). See SMITH (D. E.).

GOTHE (G.). Raumlehre für Knabenschulen. Leipzig, Freytag, 1912. 8vo. 80 pp.

HAWKES (H. E.) and others. Complete school algebra. Teacher's edition. Boston, Ginn, 1912. 12mo. 3+486 pp. Cloth. $1.75

HEATH (R. S.). A textbook of elementary trigonometry. London, Clarendon Press, 1913. 8vo. 220 pp. 3s. 6d.

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LENNES (N. J.). See SLAUGHT (H. E.).

LINK (T.). See VOLLKOMMER (M.).

MACMILLAN. Logarithmic and trigonometric tables. Prepared under the direction of E. R. Hedrick. New York, Macmillan, 1913. 8vo. 17+124 pp. Cloth.

METZLER (W. H.). See SMITH (E. R.).

ROBINSON (J. W.). Robinsonian multiplication and division tables. 2d edition. New Orleans, 1912. Folio. 108 pp. Cloth. $3.50

SCHUR (F.). Lehrbuch der analytischen Geometrie. 2te, vermehrte Auflage. Leipzig, 1912. 8vo. M. 6.50

SLAUGHT (H. E.) and LENNES (N. J.). First principles of algebra. Complete course. Boston, Allyn and Bacon. 12mo. 492 pp. Half leather. $1.20

SMITH (D. E.) and GOLDZIHER (C.). Bibliography of the teaching of mathematics, 1900–1912. (Bulletin of the United States Bureau of Education.) Washington, D. C., Government Printing Office, 1913. 8vo. 95 pp. Paper.

SMITH (E. R.) and METZLER (W. H.). Solid geometry developed by the syllabus method. New York, American Book Co., 1913. 12mo. 12 +403 pp. Cloth. $0.75

SUPPANTSCHITSCH (R.). Lehrbuch der Arithmetik und Algebra. Wien, Tempsky, 1913. 8vo. 331 pp. Cloth. M. 4.00

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III. APPLIED MATHEMATICS.

AMAGAT (E. H.). Notes sur la physique et la thermodynamique. Paris, Dunod et Pinat, 1912. 8vo. Fr. 6.00

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BLOCK (W.). Masse und Messen. (Aus Natur und Geisteswelt.) Leipzig, Teubner, 1912. 8vo. 4+111 pp. M. 1.25

BLONDEL (A. E.). Synchronous motors and convertors. Translated from the French by C. O. Mailloux. With additional chapters by C. A. Adams. New York, McGraw-Hill, 1913. 8vo. 300 pp. Cloth. $3.00

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GOLDHAMMER (D. A.). Dispersion und Absorption des Lichtes in ruhenden isotropen Körpern. Theorie und ihre Folgerungen. (Mathematisch-Physikalische Schriften für Ingenieure und Studierende. Nr. 16.) Leipzig, Teubner, 1913. 8vo. 6+144 pp. Cloth. M. 4.00

GRUNER (P.). Elementare Darlegung der Relativitätstheorie. 2te Auflage. Bern, 1913. M. 0.80

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KEMPSE (H. R.). The engineer's year book of formulae, rules, tables, data and memoranda, for 1913. London, Lockwood, 1913. 8vo. 1588 pp. Leather. 12s. 6d.

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MEHRTENS (G. C.). Vorlesungen über Ingenieur-Wissenschaften. Teil I, Band III, 1te Hälfte: Gewölbe und Stützmauern. 2te umgearbeitete und stark vermehrte Auflage. Leipzig, Engelmann, 1912. Cloth. M. 9.50

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MÜLLER-BRESLAU (H.). Die graphische Statik der Baukonstruktionen. 1ter Band. 5te, vermehrte Auflage. Leipzig, Kröner, 1912. Half leather. M. 22.00

NEUDECK (G.), SCHULZE (B.) und BLOCHMANN (R.). Der moderne Schiffbau. 1ter Teil: Geschichtliche Entwicklung des Schiffes, theoretischer und praktischer Schiffbau. Leipzig, Teubner, 1912. 8vo. M. 9.00

PEABODY (C. H.) and MILLER (E. F.). Steam-boilers. 3d edition. New York, Wiley, 1913. 8vo. 543 pp. Cloth. $4.00

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RAMSEY (A. S.). A treatise on hydromechanics. Part II: Hydrodynamics. London, Bell, 1913. 8vo. 374 pp. 10s. 6d.

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SILVESTRE (E.). See BONHOMME (J.).

SMITH (C. A. M.) and WARREN (A. G.). The new steam tables, together with their derivation and application. London, Constable, 1913. 8vo. 114 pp. 4s.

STOÏANOVITCH (C.). Explications des phénomènes physiques et sociaux. Belgrade, 1910. 8vo. 288 pp.

——. Leçons de mécanique professées a l'Université de Belgrade. Belgrade, 1912. 8vo. 18+470 pp.

TIMBLE (W. H.). Essentials of electricity. London, Chapman & Hall, 1913. 8vo. 5s. 6d.

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WARREN (A. G.). See SMITH (C. A. M.).

WEISS (M.). Die geschichtliche Entwicklung der Photogrammetrie und die Begründung ihrer Verwendbarkeit für Mess- und Konstruktionszwecke. Stuttgart, Strecker und Schröder, 1913. 8vo. 7+94 pp. M. 14.00

WILDA (H.). Steam turbines. Their theory and construction. Translated from the German by C. Salter. New York, Van Nostrand, 1913. 12mo. 12+191 pp. Cloth. $1.25

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